PAnDAG Workshop, 15~16/05/2025

The "phoenix" and "fork anywhere" models of git graphs

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MOTIVATION

"DAGs are too broad; feature branch git graphs could be broadened."

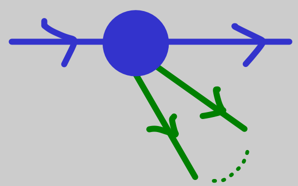
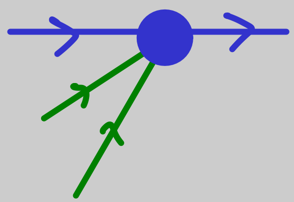
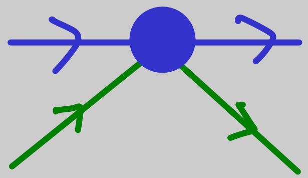
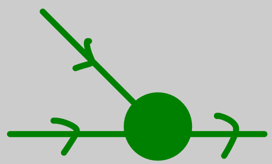
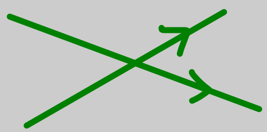
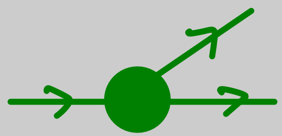
→ Come up with superclasses of feature branch git graphs, such that:

* they still describe realistic git workflows;
* their enumeration / random generation are not too difficult.

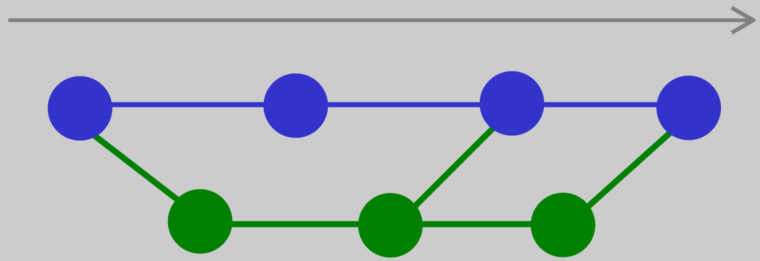
STARTING POINT

Reminder

Feature branch graphs rules:

OK	OKn't
	
	
	

Broaden this class of graphs by allowing feature branches to be "reborn" when they go to "die" on the main branch,

e.g. 

PHOENIX GRAPHS

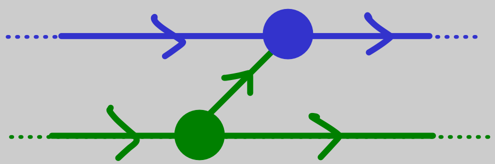
Definition

Phoenix graph = DAG with :

* a linear main branch
* zero or more feature branches (DAGs starting on a main vertex, and ending on one or more feature vertices further to the right)

* indegree($\bullet$) $\in \{1,2\}$ ($=0$ for leftmost)
* outdegree($\bullet$) ≥ 1 ($=0$ for rightmost)
* indegree($\bullet$) $= 1$
* outdegree($\bullet$) $\in \{1,2\}$, $=2$ iff one child of each colour

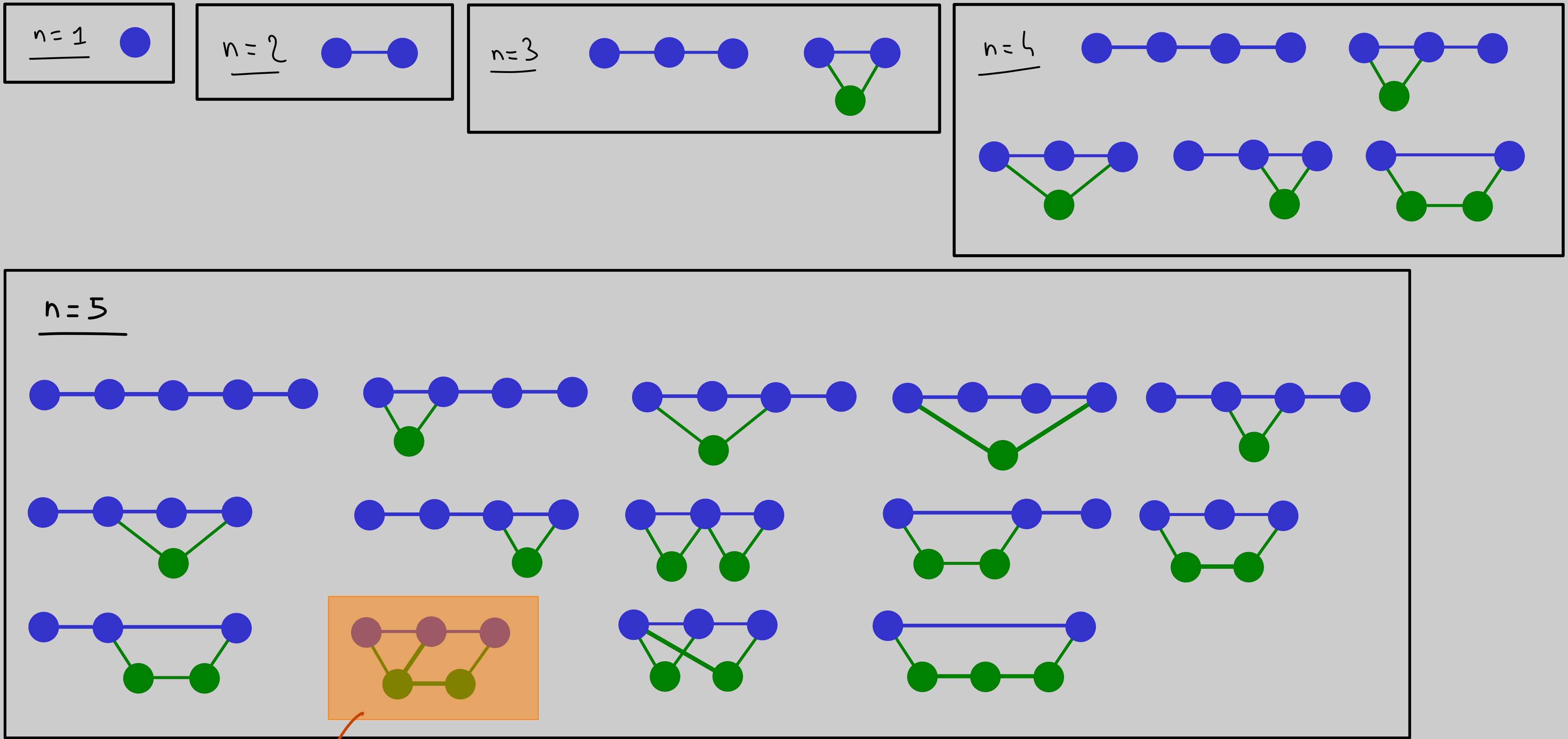
i.e. a feature branch graph, but the pattern



is allowed.

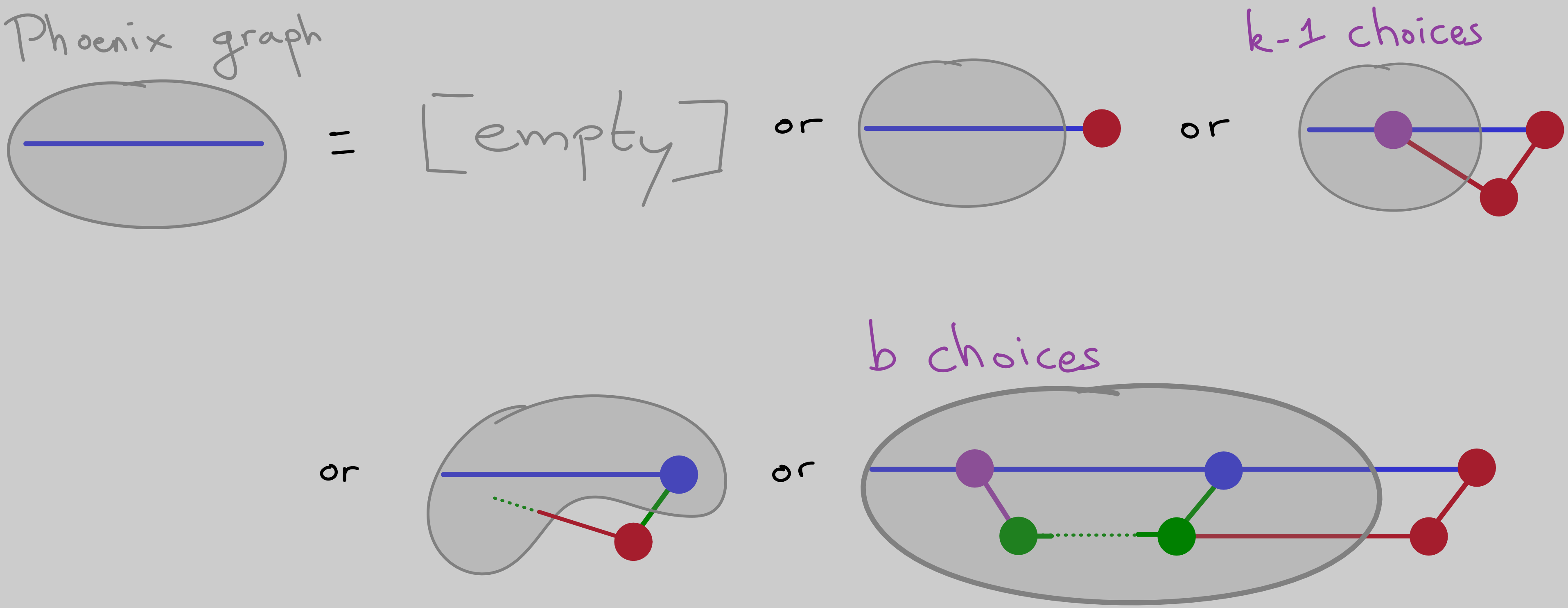
EXAMPLES

Size n of a phoenix graph = number of vertices



smallest non feature branch phoenix graph

RECURSIVE DECOMPOSITION



$$g_{n,k,b} = g_{n-1,k-1,b} + (k-1)\, g_{n-2,k-1,b-1}$$
$$+ \left(g_{n-1,k,b} - g_{n-2,k-1,b}\right) + b \cdot g_{n-2,k-1,b}$$

number of phoenix graphs with n vertices, k main vertices, b feature branches

GENERATING FUNCTION

$\star$ $g_{n,k,b} = g_{n-1,k-1,b} + (k-1) g_{n-2,k-1,b-1} + g_{n-1,k,b} + (b-1) g_{n-2,k-1,b}$

$\star$ $\tilde{G}(z,u,v) := \sum_{n,k,b \geq 0} g_{n,k,b} \, z^n u^k v^b$ ["$\tilde{G}$" for short]

↳ not analytic...

GENERATING FUNCTION

* $g_{n,k,b} = g_{n-1,k-1,b} + (k-1) g_{n-2,k-1,b-1} + g_{n-1,k,b} + (b-1) g_{n-2,k-1,b}$

* $\widetilde{G}(z,u,v) := \sum_{n,k,b \geq 0} g_{n,k,b}\, z^n \frac{u^k}{k!} \frac{v^b}{b!}$ ["$\widetilde{G}$" for short]

$\hookrightarrow$ $z^2 \widetilde{G} - z^2 u \partial_u \widetilde{G} - z \partial_v \widetilde{G} + (1-z) \partial_u \partial_v \widetilde{G} - z^2 v \partial^2_{v,v} \widetilde{G} = 0$

Sadly, no solution :(

Moreover, the sequence $\left(\sum_{k,b} g_{n,k,b}\right)_{n \geq 0}$ does not appear in the OEIS.

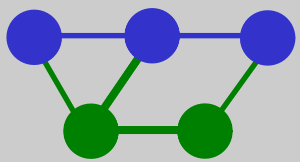
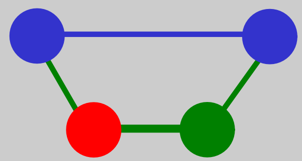
$\longrightarrow$ What else *can* we say about phoenix graphs?

BABY PHOENIXES

Definition

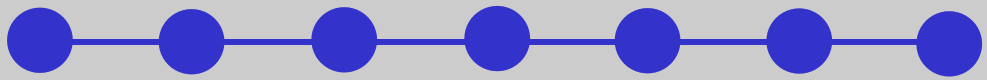
Baby phoenix graph = phoenix graph where every feature vertex is part of a merge (i.e. one of its children is a main vertex, and no main vertex has its indegree equal to 1

Example

 is a baby phoenix, but  is not.

HOW ARE BABIES MADE?

To construct a baby phoenix graph with $k \geq 1$ main vertices, start with the main branch:



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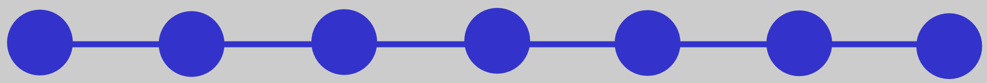
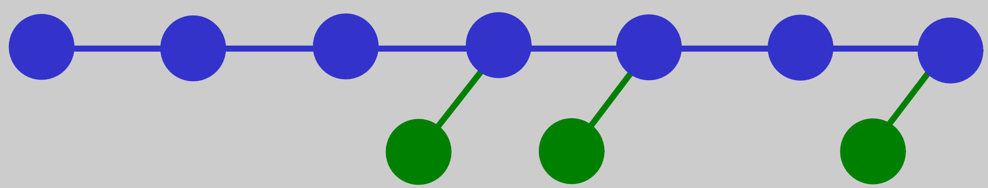


Then, among the $k-1$ rightmost vertices, take a nonempty subset, make each one of them part of a merge (adding feature vertices as needed), make them part of the same feature branch, and finally choose where that branch starts:



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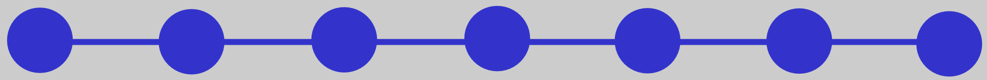

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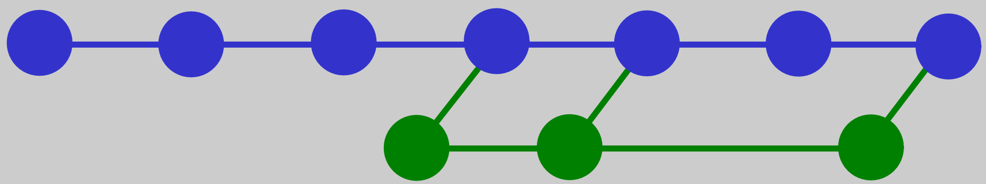

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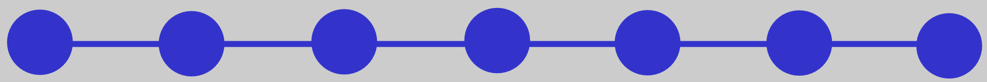

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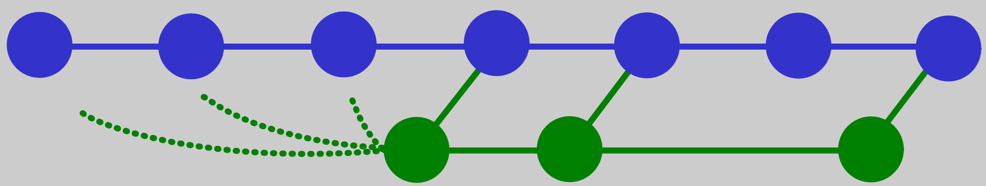

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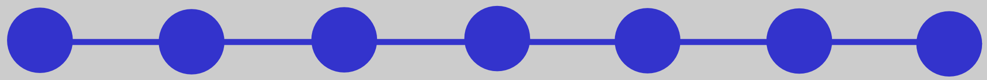

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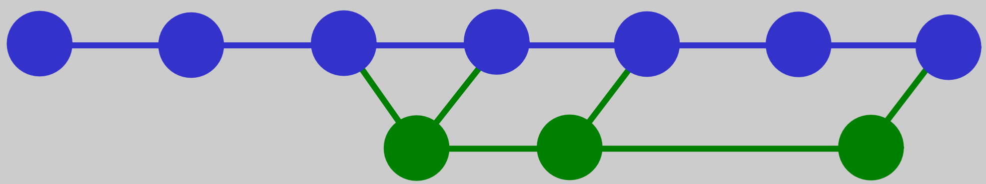
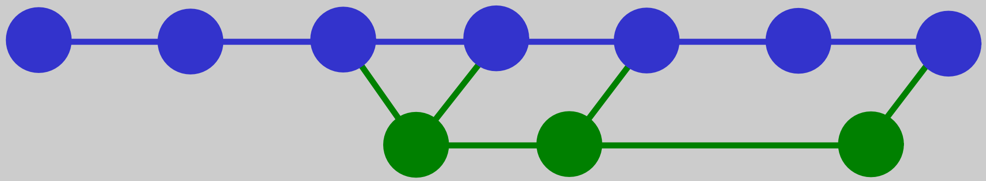


HOW ARE BABIES MADE?

Repeat this process with the available main vertices to add other feature branches:



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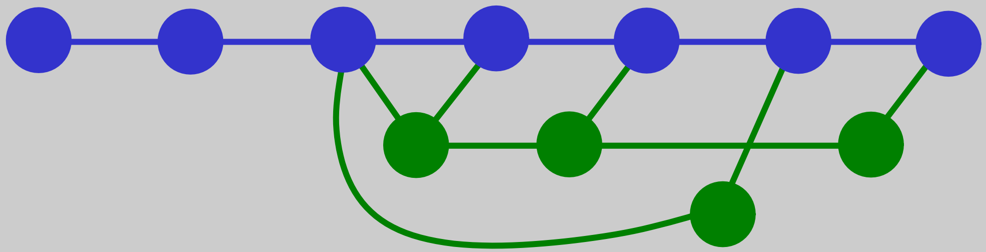

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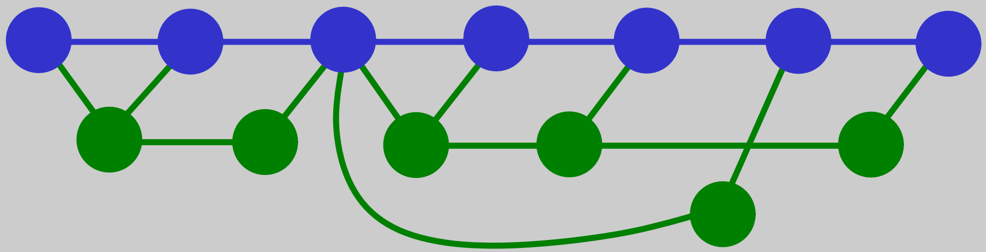

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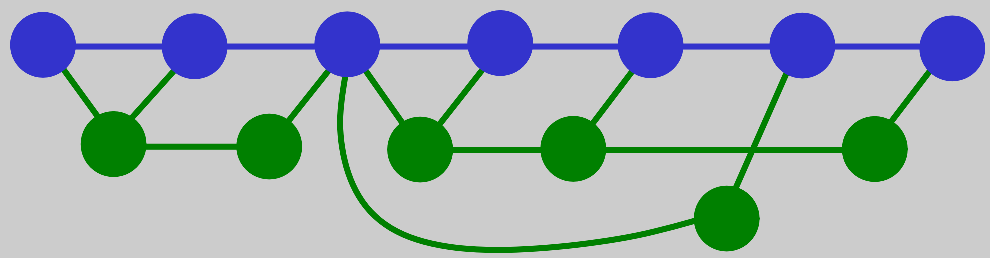

Notes

- A baby phoenix graph with $k \geq 1$ main vertices necessarily has $k-1$ feature vertices.
- Counting sequence: 1, 1, 3, 14, 89, 716, ...

  ⟶ OEIS A007549

BIJECTION WITH SET PARTITIONS

Labelling the main vertices from 0 to $k-1$, we can map a baby phoenix graph with k main vertices to a set partition of size $k-1$:

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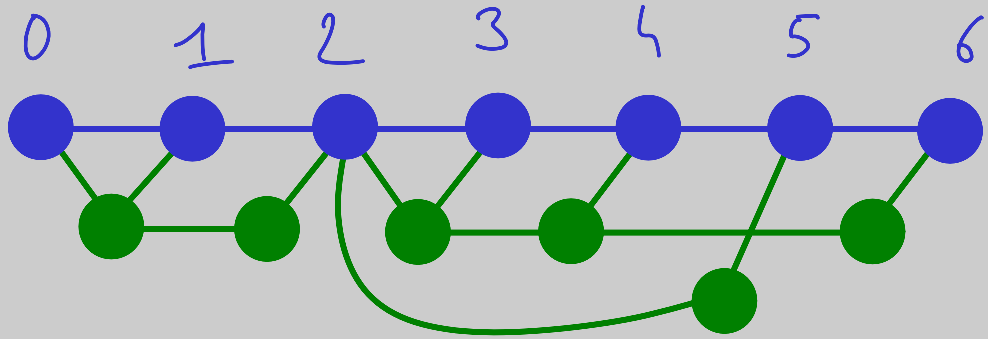

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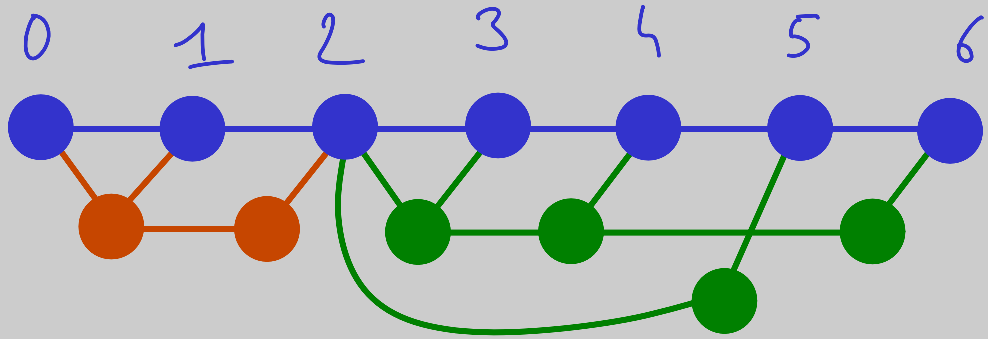

$\{1, 2\}$

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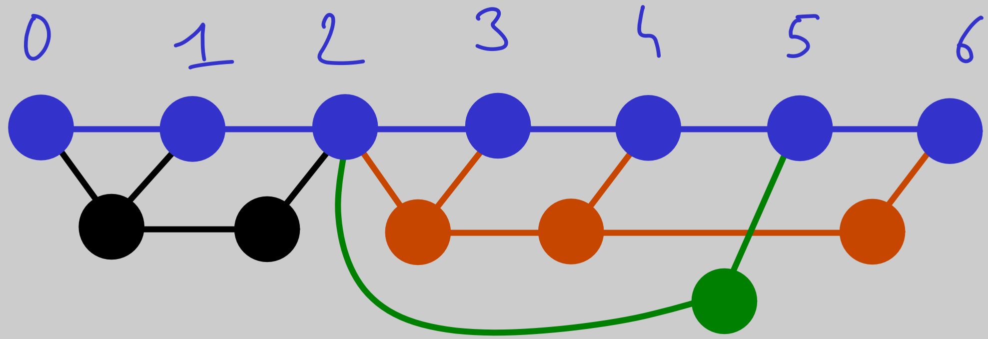

$\{1,2\}\ \{3,4,6\}$

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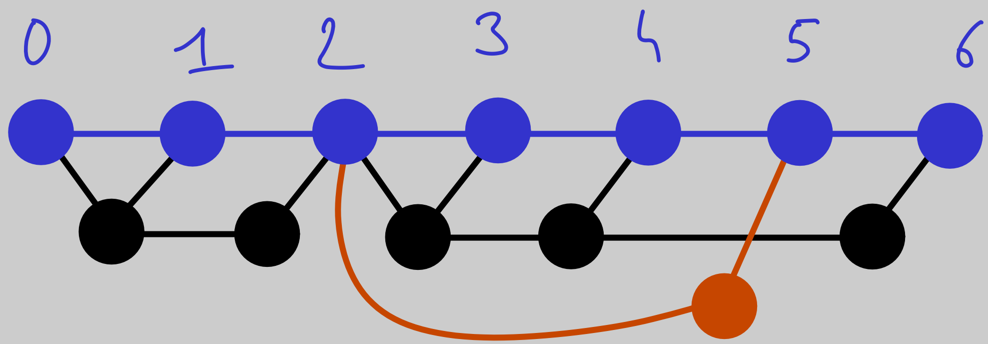

$\{1,2\}\ \{3,4,6\}\ \{5\}$

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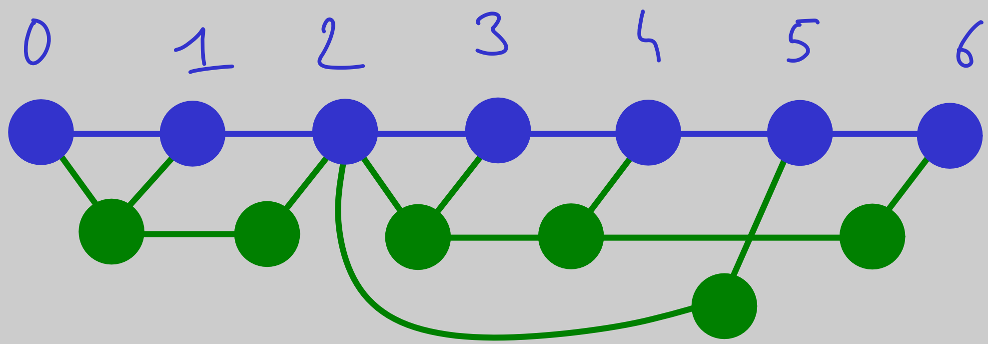

$$[6] = \{1,2\} \sqcup \{3,4,6\} \sqcup \{5\}$$

$\hookrightarrow$ This is a surjection. How do we make it a bijection?

BIJECTION WITH SET PARTITIONS

How many ways are there to obtain a given part $\{k_1, k_2, \dots, k_m\}$ (here, $k_1 < k_2 < \dots < k_m$)?

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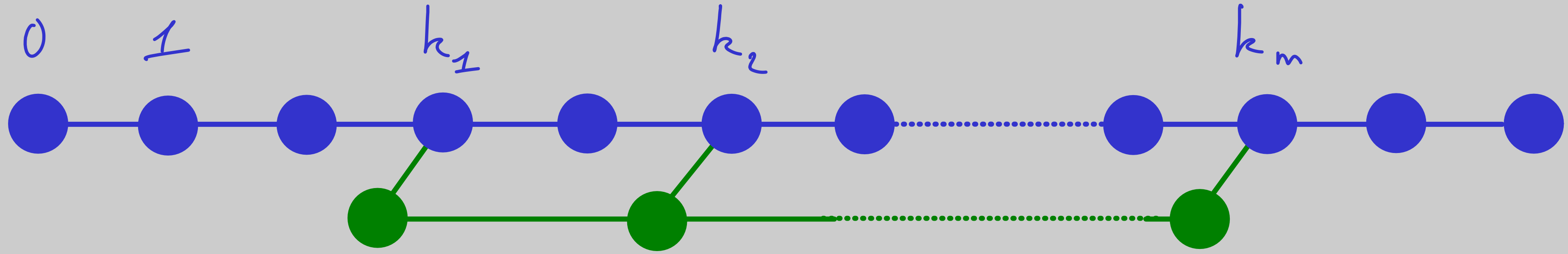

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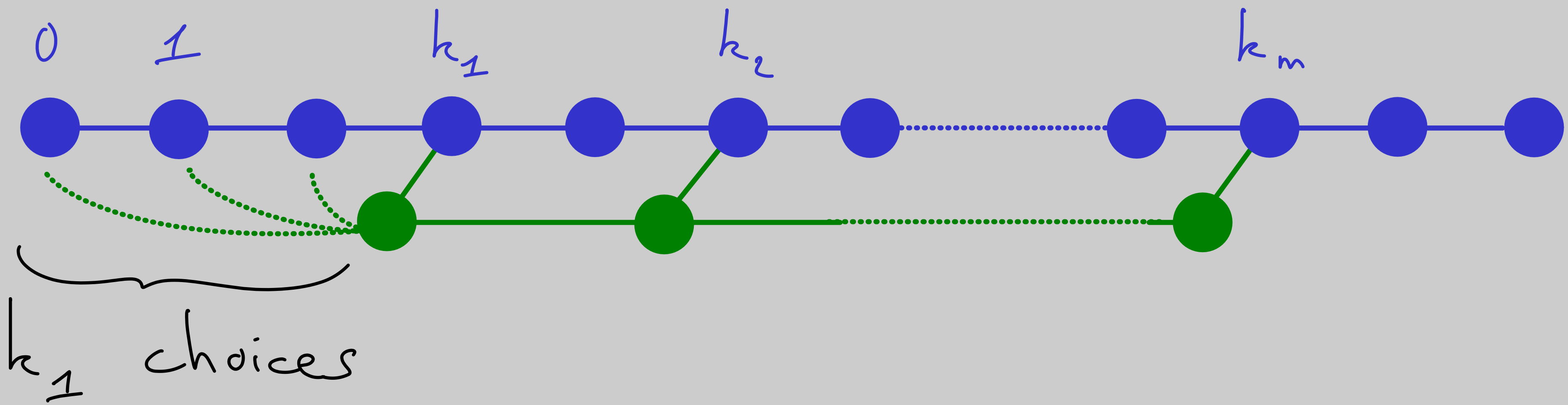

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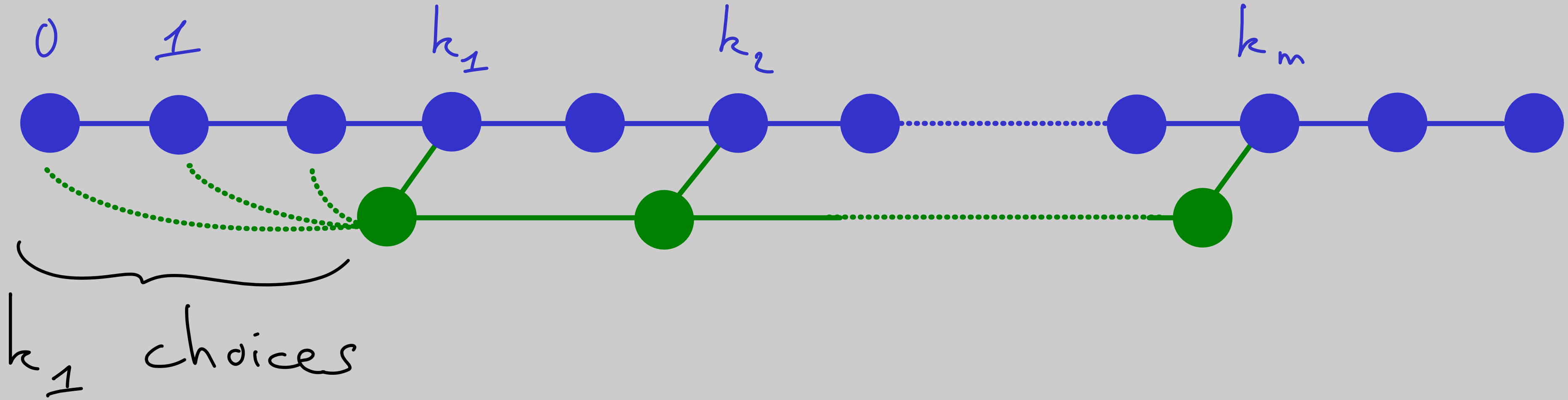


$\hookrightarrow$ Baby phoenix graphs with k main vertices

$$\simeq$$

Set partitions of size $k-1$,

weighted by the product of part minima

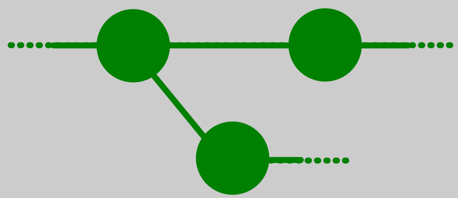
FORK ANYWHERE

Definition

["FA" for short]

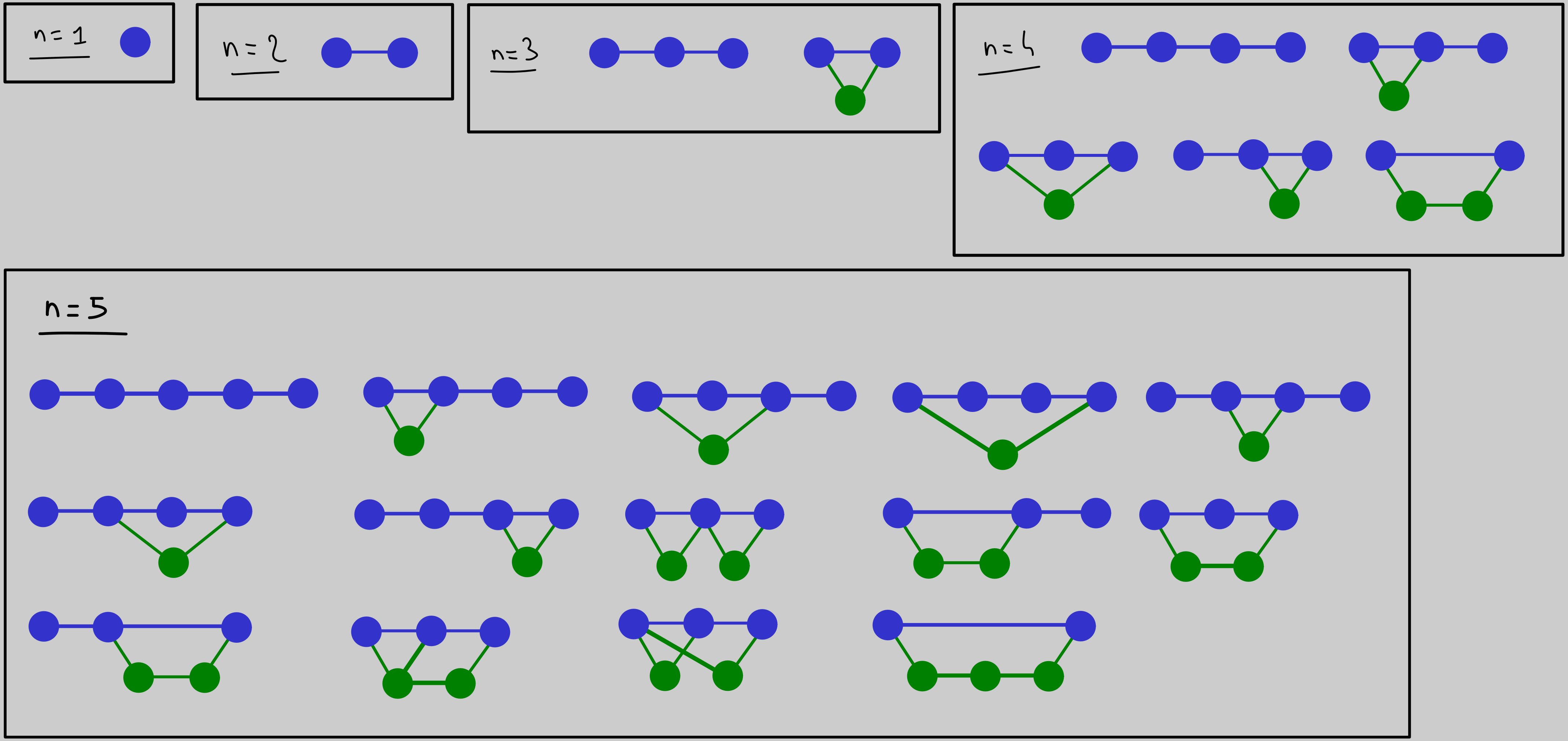
Fork anywhere graph = phoenix graph, but vertices on feature branches can have their outdegree equal to 2 without restriction on their children's types,

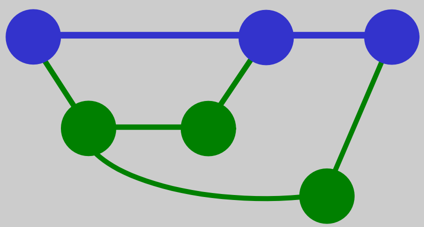
i.e. a phoenix graph, but the pattern



is allowed.

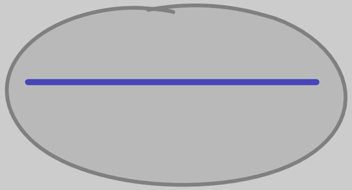
EXAMPLES



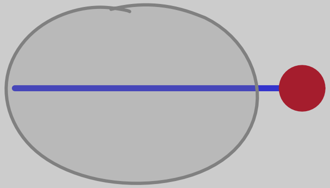
 is the smallest non phoenix FA graph.

RECURSIVE DECOMPOSITION

FA graph



= [empty] or



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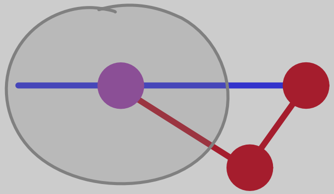

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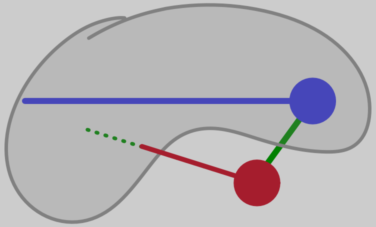

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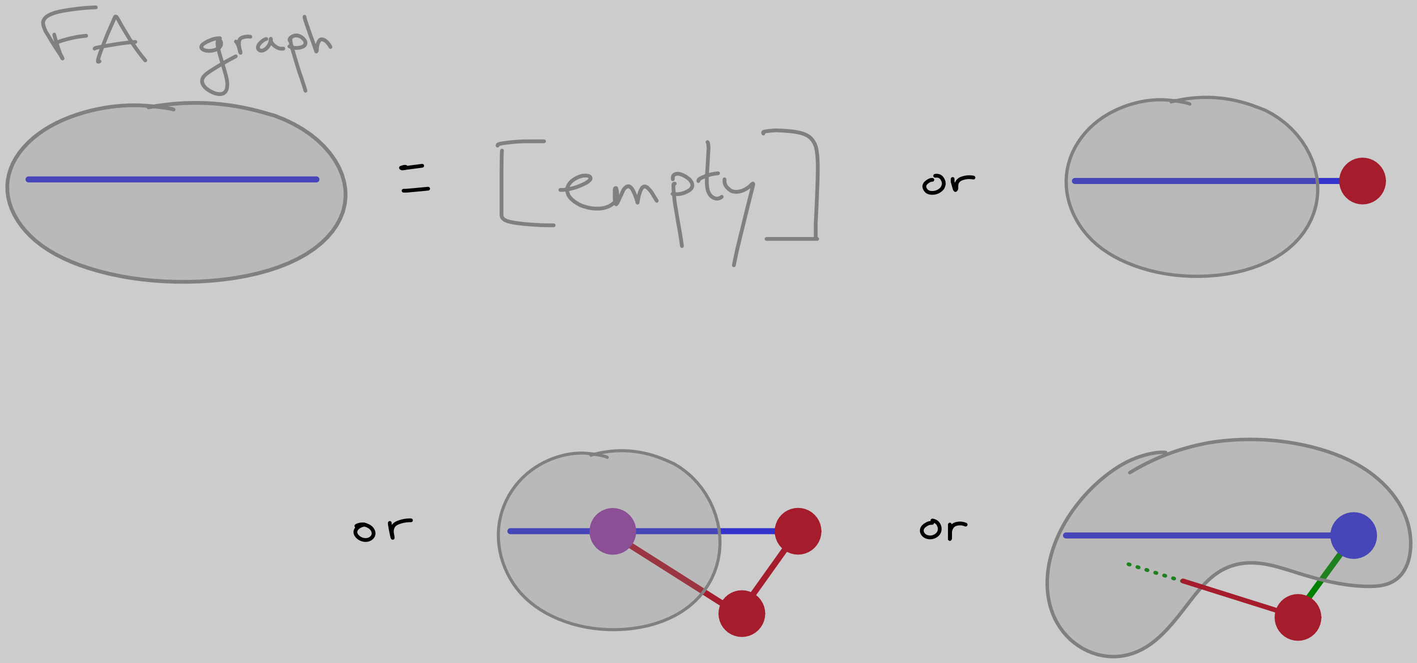

OGF "G":

$$G(z) = \sum_{n \geq 0} g_n z^n$$

EGF "$\tilde{G}$":

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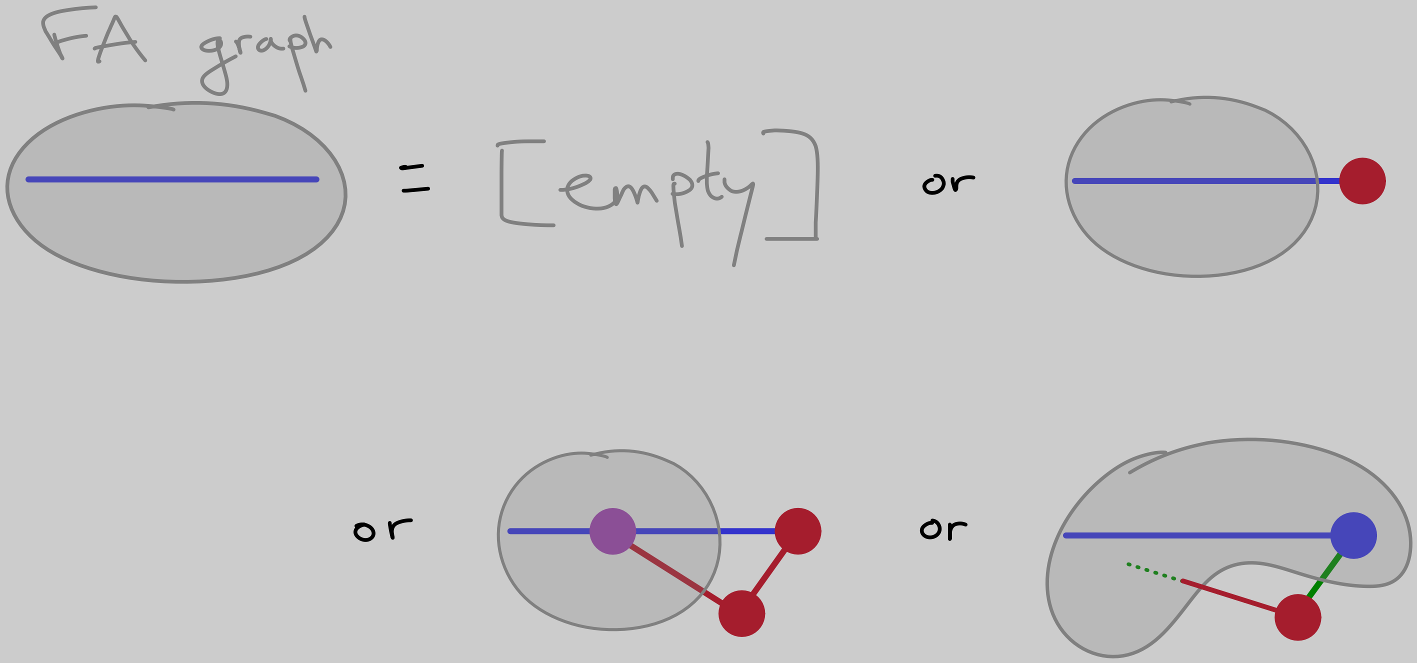

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$$\hookrightarrow G(z) = 1 + zG(z) + z^2 \cdot zG'(z) + z \cdot \left(G(z) - zG(z) - 1 \right)$$

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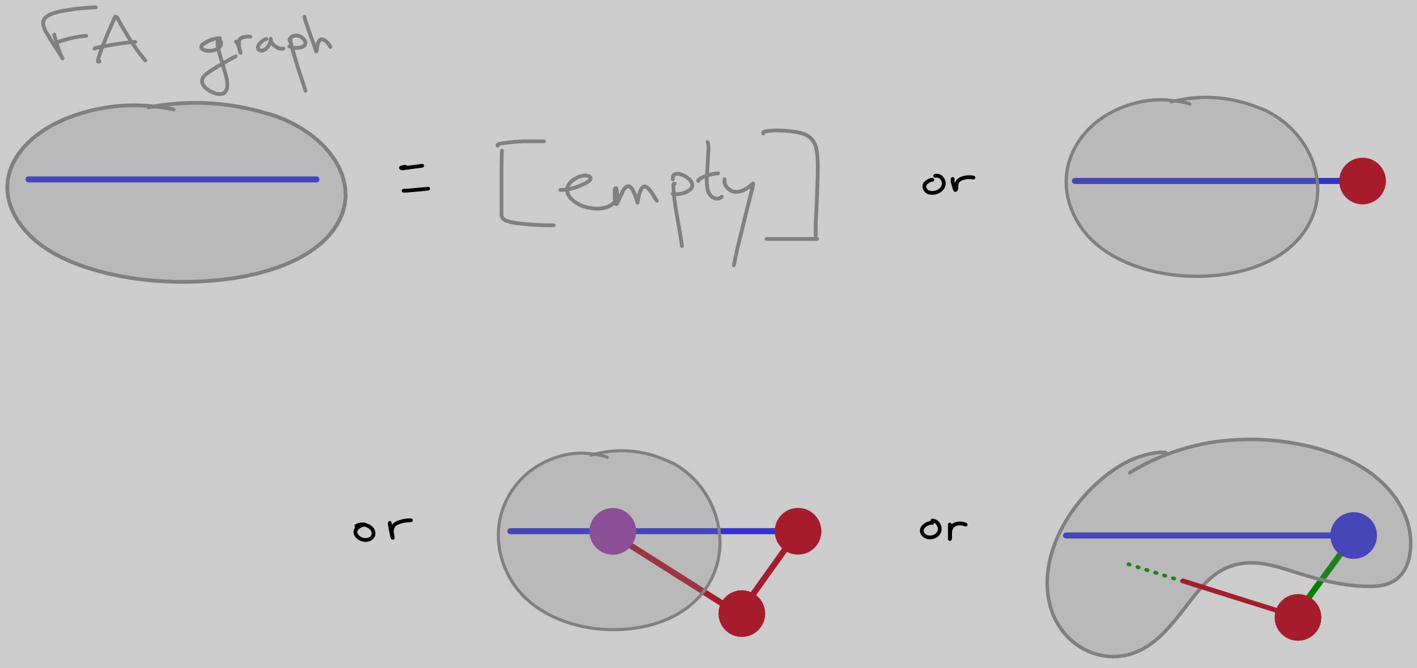

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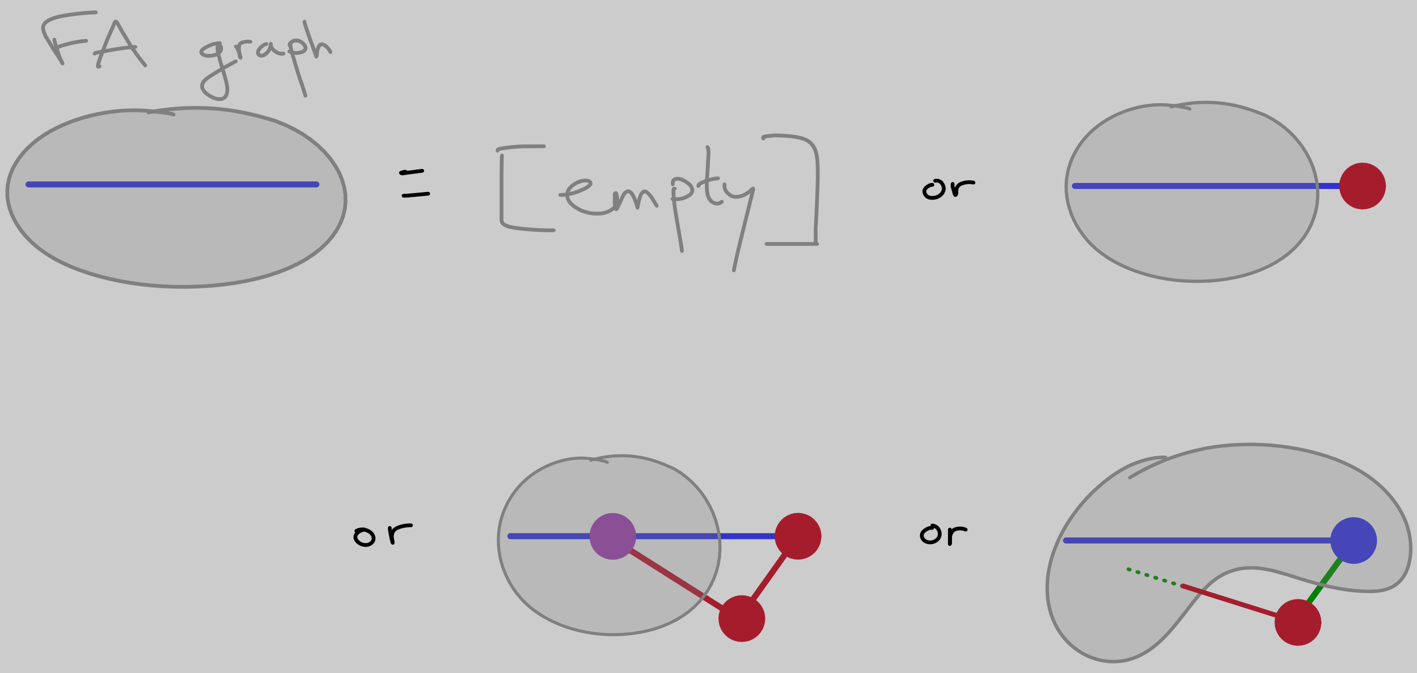

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$\longrightarrow$ $\dfrac{z^2}{2}\tilde{G}''(z) = \dfrac{z^2}{2} e^{z} e^{z + z^2/2}$

$\hookrightarrow$ $\hat{G} := \tilde{G}''$ satisfies $\hat{G}(z) = \exp\left(2z + \frac{1}{2}z^2\right)$

$$\hat{G}(z) = \overbrace{\exp\left(2z + \frac{1}{2}z^2\right)}^{GF}$$

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$\hookrightarrow$ symbolic: $\mathcal{G} = \mathrm{SET}\left(\mathcal{Z}_\bullet \sqcup \mathcal{Z}_\circ \sqcup \mathrm{CYC}_2(\mathcal{Z}_\circ)\right)$

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↳ can be interpreted as involutions with 2 types of fixed points

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↳ asymptotics : GF has no singularity

GF

$$\hat{G}(z) = \exp\left(2z + \frac{1}{2}z^2\right)$$

↳ symbolic : $\mathcal{G} = \mathrm{SET}(\mathcal{Z}_\bullet \sqcup \mathcal{Z}_\circ \sqcup \mathrm{CYC}_2(\mathcal{Z}_\circ))$

↳ can be interpreted as involutions with 2 types of fixed points

↳ Bijection ?

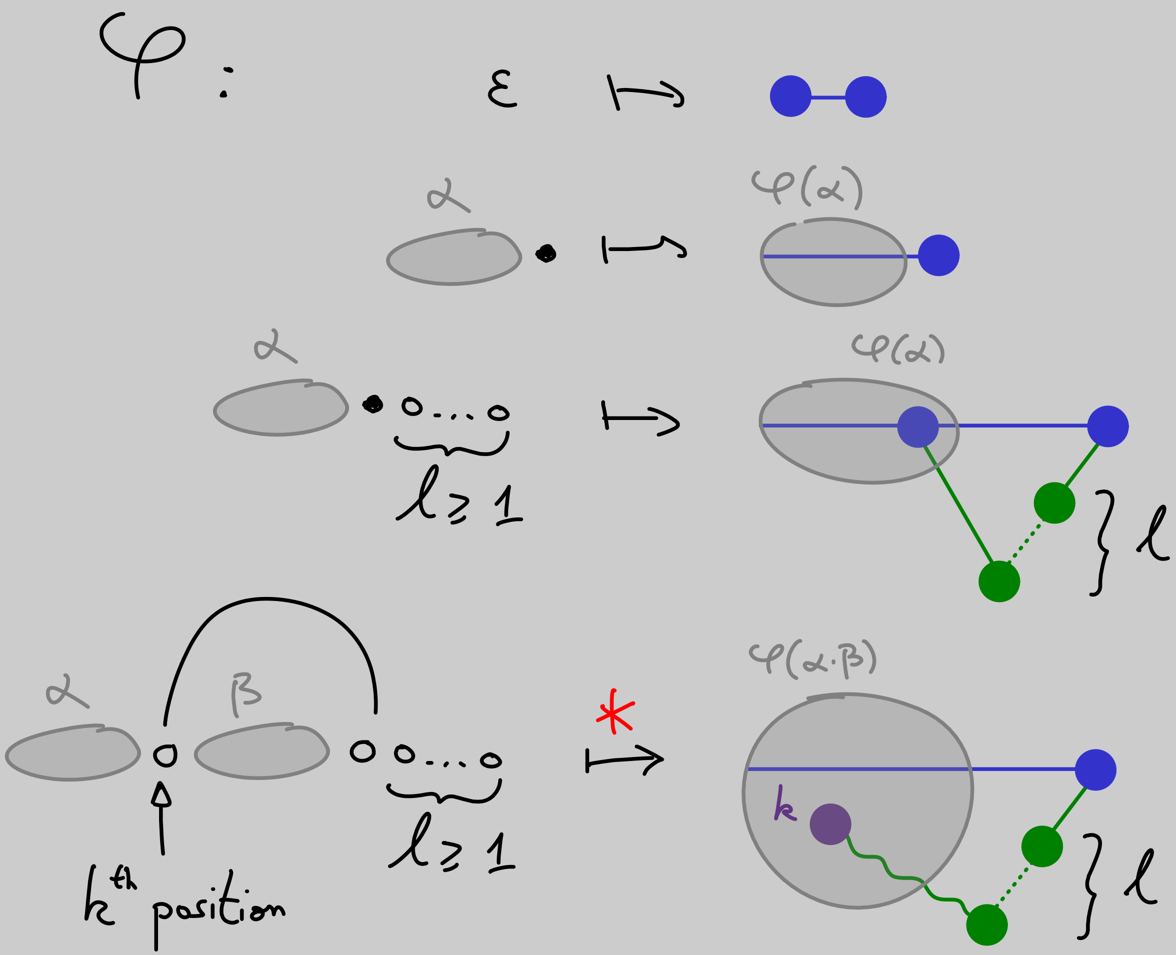
↳ Random generation of FA ?

↳ asymptotics : GF has no singularity

↳ Saddle point method ?

BIJECTION(S)

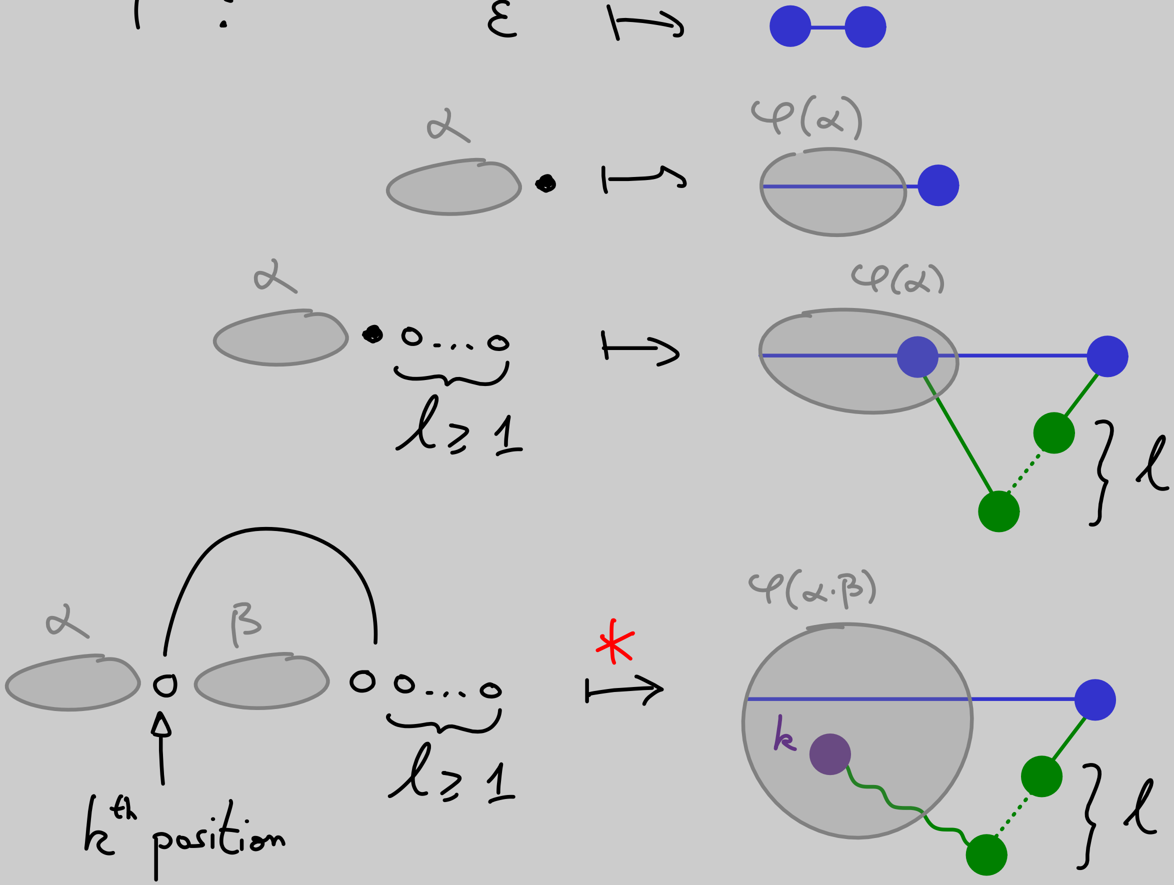
Example



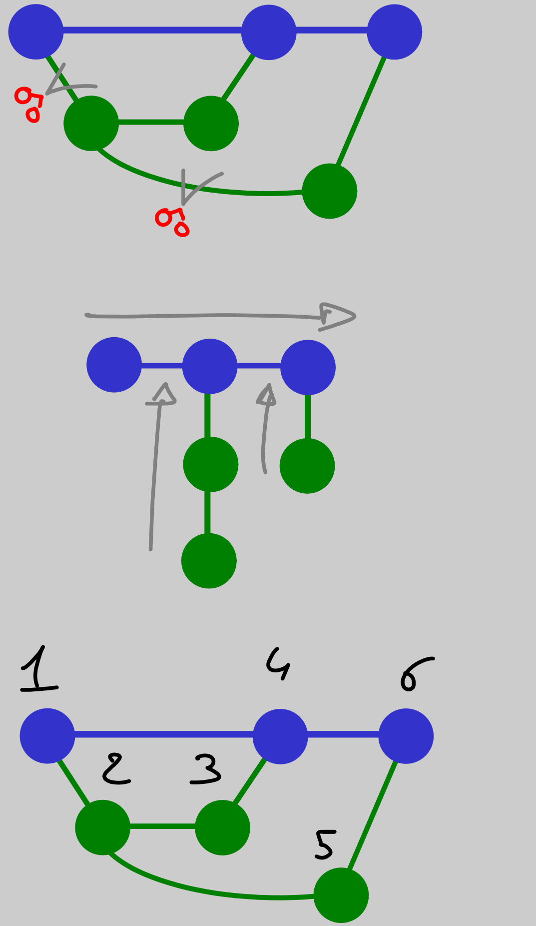
BIJECTION(S)

Example

φ :



* FA graph vertex labelling :



BIJECTION(S)

Open problem

feature branch $\subset$ phoenix $\subset$ fork anywhere

$\big\downarrow$ a certain bijection

involutions with
2 types of fixed points

BIJECTION(S)

Open problem

$$\text{feature branch} \subset \text{phoenix} \subset \text{fork anywhere}$$

same bijection | same bijection | a certain bijection

$$? \subset ? \subset \text{involutions with 2 types of fixed points}$$

BIJECTION(S)

Open problem

$$\begin{array}{ccccc}
\text{feature branch} & \subset & \text{phoenix} & \subset & \text{fork anywhere} \\
\big\downarrow \text{same bijection} & & \big\downarrow \text{same bijection} & & \big\downarrow \text{a certain bijection} \\
? & \subset & ? & \subset & \text{involutions with 2 types of fixed points}
\end{array}$$

Ideally, we would want the ?'s to be subclasses defined by pattern avoidance.

RANDOM GENERATION

Boltzmann sampler for involutions :

$$\forall x > 0, \quad \mathbb{P}_x(i) = \frac{x^{|i|}}{|i|! \cdot \hat{G}(x)}$$

involution

RANDOM GENERATION

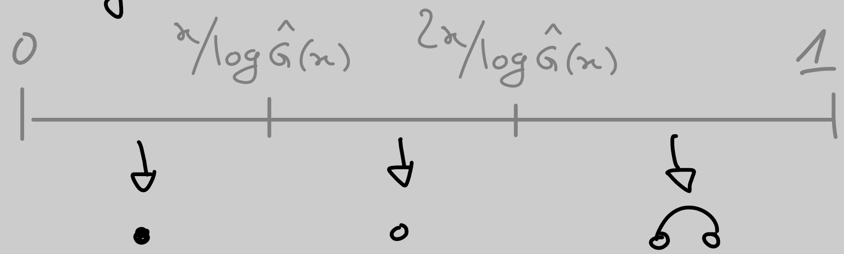
Boltzmann sampler for involutions :

$$\forall x > 0, \quad \mathbb{P}_x(\underset{\substack{\uparrow \\ \text{involution}}}{i}) = \frac{x^{|i|}}{|i|! \cdot \hat{G}(x)}$$

$$\hat{G}(x) = \exp\left(2x + \frac{1}{2}x^2\right)$$

$\hookrightarrow$ * size $\rightsquigarrow$ Poisson$\left(2x + \frac{1}{2}x^2\right)$

* draw each of the nodes according to $u \rightsquigarrow \mathcal{U}_{[0,1]}$



RANDOM GENERATION

Boltzmann → involution with 2 types of fixed points

↓

FA graph ← any working bijection

```
%%time

boltz_involution(10**6)
```

```
CPU times: user 1.58 s, sys: 37.6 ms, total: 1.62 s
Wall time: 1.62 s
```

ASYMPTOTICS

Cauchy formula: $[z^n]\,\hat{G}(z) \underset{n\to\infty}{\sim} \frac{1}{2i\pi} \oint \frac{\hat{G}(z)}{z^{n+1}}\, dz$

ASYMPTOTICS

Cauchy formula: $[z^n]\,\hat{G}(z) \underset{n\to\infty}{\sim} \underbrace{\frac{1}{2i\pi}\oint \frac{\hat{G}(z)}{z^{n+1}}\,dz}_{\substack{\text{approximation given} \\ \text{by saddle point method}}}$

ASYMPTOTICS

Cauchy formula: $[z^n]\,\hat{G}(z) \underset{n\to\infty}{\sim} \underbrace{\frac{1}{2i\pi}\oint \frac{\hat{G}(z)}{z^{n+1}}\,dz}$

approximation given by saddle point method

↳ ugly approx.:

$$[z^n]\,\hat{G}(z) \underset{n\to\infty}{\sim} \frac{\exp\left(\frac{1}{2}\left(n+2\sqrt{n+1}-2\right)\right)}{2\sqrt{\pi\left(n+1-\sqrt{n+1}\right)}\cdot\left(\sqrt{n+1}-1\right)^n}$$

OPEN QUESTIONS

* Study other classes of git graphs (realism, ease of computation, ...)
* Evaluate how "close" a typical DAG is to all of these git graph classes

OPEN QUESTIONS

- ☆ Study other classes of git graphs (realism, ease of computation, ...)
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Thank you :)