An introduction to Dyck paths with air pockets

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Dyck paths with air pockets, $\mathcal{A}_n$

A *Dyck path with air pockets* is a non empty lattice path in the first quadrant of $\mathbb{Z}^2$ starting at the origin, ending on the x-axis, and consisting of up-steps $U = (1,1)$ and down-steps $D_k = (1,-k)$, $k \geqslant 1$, where two down steps cannot be consecutive (we set $D = D_1$ for short). The set of such paths is denoted by $\mathcal{A}$.

Let $\mathcal{A}_n$ denote the set of n-length Dyck paths with air pockets.

Example:

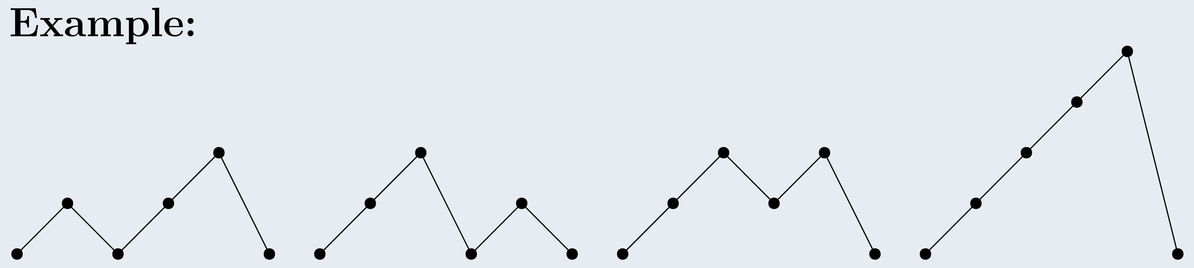


Figure: $\mathcal{A}_5 = \{UDU^2D_2, U^2D_2UD, U^2DUD_2, U^4D_4\}$.

Cardinality formula (OEIS A004148):

$$|\mathcal{A}_n| = \sum_{k=\lceil n/2 \rceil}^{n} \frac{1}{k}\binom{k}{n-k}\binom{k}{n-1-k}.$$

Generating function:

$$\sum_{n=0}^{\infty} |\mathcal{A}_n| \cdot x^n = \frac{1 + x - x^2 - \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}}{2x}.$$

Taylor expansion:

$$1 + x^2 + x^3 + 2x^4 + 4x^5 + 8x^6 + 17x^7 + 37x^8 + 82x^9 + 185x^{10} + \dots$$

A Dyck path with air pockets is *prime* if it ends with D_k, $k \geqslant 2$, and it returns to the x-axis only once. The set of such paths is denoted by $\mathcal{P}$.

Lowered and elevated paths

We introduce two transformations of Dyck paths with air pockets. If α is a Dyck path with air pockets of the form $U\beta D_k$ (where β is either empty or in $\mathcal{A}$), then we define the *elevated* version of α as $\alpha^\sharp = U^2\beta D_{k+1}$. We also define the inverse operation $\cdot^\flat$, and call $\alpha^\flat$ the *lowered* version of α.

Example:

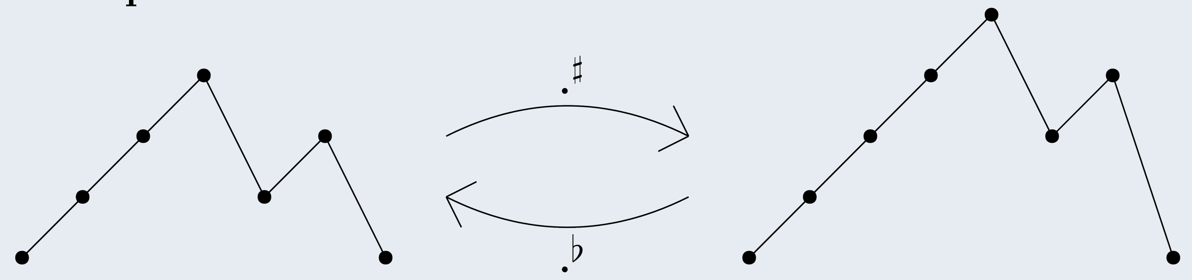


Figure: The paths $U^3D_2UD_2$ (left) and $U^4D_2UD_3$ (right) are lowered and elevated versions of one another.

The operations $\cdot^\flat$ and $\cdot^\sharp$ will help us to define a bijection between $\mathcal{A}_n$ and a class of well-known lattice paths.

Current works and goals

Study of Dyck paths with air pockets which avoid certain patterns (i.e. U^3, DUD, ...), in search of a general method for finding the enumeration of such subsets of $\mathcal{A}_n$.

Example ($\mathcal{A}_n(U^3)$):

$$1 + x^2 + x^3 + x^4 + 3x^5 + 3x^6 + 6x^7 + 11x^8 + 15x^9 + 31x^{10} + \dots$$

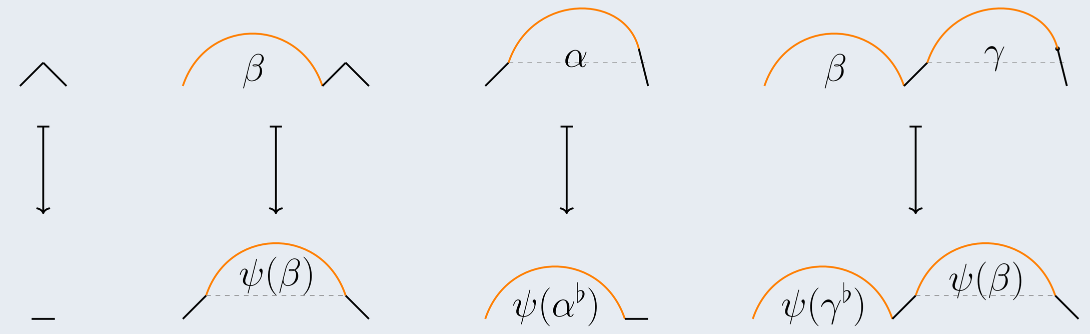
Is there a way to obtain this result with a general algorithm, rather than studying $\mathcal{A}_n(U^3)$ as its own thing?

Bijection with peakless Motzkin paths, $\mathcal{M}_n$

A *peakless Motzkin path* is a non empty lattice path in the first quadrant of $\mathbb{Z}^2$ starting at the origin, ending on the x-axis, consisting of up-steps $U = (1,1)$, down-steps $D = (1,-1)$, and flat-steps $F = (1,0)$, having no occurrence of UD. The set of such paths is denoted by $\mathcal{M}$, and the subset of n-length peakless Motzkin paths is denoted by $\mathcal{M}_n$.

$$\mathcal{A} \xrightarrow{\psi} \mathcal{M}$$

$$\alpha \overset{\psi}{\longmapsto} \begin{cases} F & \alpha = UD \\ U\psi(\beta)D & \alpha = \beta UD, \beta \in \mathcal{A} \\ \psi(\alpha^\flat)F & \alpha \in \mathcal{P} \\ \psi(\gamma^\flat)U\psi(\beta)D & \alpha = \beta\gamma, \beta \in \mathcal{A}, \gamma \in \mathcal{P} \end{cases}$$



Theorem: The map ψ induces a bijection between $\mathcal{A}_n$ and $\mathcal{M}_{n-1}$.

Pattern popularity generating function example

For all $k \geqslant 1$, the g.f. for the total number of occurrences of $U^\ell D_\ell$, $\ell \geqslant k$, in $\mathcal{A}_n$ is:

$$\frac{x^{k+1}\left(1 + 2x^2 - x^3 + (1-x)\sqrt{(x^2+x+1)(x^2-3x+1)}\right)}{2(1-x)\sqrt{(x^2+x+1)(x^2-3x+1)}}.$$

(OEIS A201631)

Transport of consecutive patterns

Dyck path with air pockets	Peakless Motzkin path
$\mathbf{U}$	$\mathbf{F + U = F + D}$
$\mathbf{D = UD}$	$\mathbb{1}_\mathbf{F} + \mathbf{UFD} + \mathbb{1}_{\mathbf{U\mathcal{M}D}} + \mathbf{U^2\mathcal{M}D^2}$
$\mathbf{DU}$	$\mathbf{UFD + U^2\mathcal{M}D^2}$
$\mathbf{UU}$	$\mathbf{F - \hat{1}}$
$\mathbf{\Delta_k}$	$\mathbb{1}_{\mathbf{F^k}} + \mathbf{UF^kD} + \mathbb{1}_{\mathbf{F^{k-1}U\mathcal{M}D}} + \mathbf{UF^{k-1}U\mathcal{M}D^2}$
$\mathbf{Peak}$	$\mathbf{U + \hat{1}}$
$\mathbf{Ret}$	$\mathbf{\hat{n} - LastF}$
$\mathbf{SLast}$	$\mathbf{Ret}$

For any path α (in $\mathcal{A}$ or in $\mathcal{M}$, depending on the relevant column in the table):

- $\mathbf{\hat{1}}(\alpha) = 1$, $\mathbf{\hat{2}}(\alpha) = 2$, $\mathbf{\hat{n}}(\alpha) = n$, and so on;
- $\mathbb{1}_\beta(\alpha)$ is 1 if $\alpha = \beta$ and 0 otherwise;
- $\mathbb{1}_{\mathbf{U\mathcal{M}D}}(\alpha)$ is 1 if there exists $\beta \in \mathcal{M}$ such that $\alpha = U\beta D$ and 0 otherwise;
- $\mathbf{U^2\mathcal{M}D^2}(\alpha)$ is the number of occurrences of $U^2\beta D^2$ in α for $\beta \in \mathcal{M}$;
- $\mathbf{\Delta_k}(\alpha)$ is the number of occurrences of U^kD_k in α;
- $\mathbf{Peak}(\alpha) = \sum_{\mathbf{k} \geqslant 1} \mathbf{UD_k}(\alpha)$;
- $\mathbf{Ret}(\alpha)$ is the number of returns to the x-axis of α;
- $\mathbf{LastF}(\alpha)$ is the position of the rightmost flat-step in α;
- $\mathbf{SLast}(\alpha)$ is the size of the the last step of α (i.e. k if the last step is D_k).

Pattern popularity in Dyck paths with air pockets

The popularity of various patterns (i.e. total number of occurrences of said patterns) in $\mathcal{A}_n$ ($2 \leqslant n \leqslant 11$) can be found in the following table:

Pattern	Pattern popularity in $\mathcal{A}_n$	OEIS
U	1, 2, 5, 13, 32, 80, 201, 505, 1273, 3217	A110320
D	1, 0, 2, 3, 7, 17, 40, 97, 238, 587	A051291
Peak	1, 1, 3, 7, 16, 39, 95, 233, 577, 1436	A203611
Ret	1, 1, 3, 6, 13, 29, 65, 148, 341, 793	A093128
Cat	0, 1, 1, 4, 8, 19, 44, 102, 239, 563	
Δ_k	$\underbrace{0,\dots,0}_{k-1 \text{ zeroes}}$, 1, 0, 2, 3, 7, 17, 40, 97, 238, 587	A051291
$\Delta_{\geqslant k}$	$\underbrace{0,\dots,0}_{k-1 \text{ zeroes}}$, 1, 1, 3, 6, 13, 30, 70, 167, 405	A201631 (u_n)
$\Delta_{\leqslant k}$	$\Delta_{\leqslant 1}$ 1, 0, 2, 3, 7, 17, 40, 97, 238, 587	$u_n - u_{n-k}$
	$\Delta_{\leqslant 2}$ 1, 1, 2, 5, 10, 24, 47, 137, 335, 825, …	
	$\Delta_{\leqslant 3}$ 1, 1, 3, 5, 12, 27, 64, 154, 375, 922, …	
	etc.	

- Cat(α) is the number of catastrophes of α, i.e. steps of the form D_k, $k \geqslant 2$, ending on the x-axis;
- $\Delta_{\geqslant k}(\alpha)$ is the number of occurrences of $U^\ell D_\ell$, $\ell \geqslant k$, in α;
- $\Delta_{\leqslant k}(\alpha)$ is the number of occurrences of $U^\ell D_\ell$, $1 \leqslant \ell \leqslant k$, in α.

We also define *non-decreasing Dyck paths with air pockets* as paths of $\mathcal{A}$ whose sequence of valley heights is non-decreasing, i.e. the sequence of the minimal ordinates of the occurrences of D_kU, $k \geqslant 1$, is non-decreasing from left to right. The set of such paths is denoted by $\mathcal{A}^{\nearrow}$, and the subset of n-length non-decreasing Dyck paths with air pockets is denoted by $\mathcal{A}_n^{\nearrow}$. We get the following table:

Pattern	Pattern popularity in $\mathcal{A}_n^{\nearrow}$	OEIS
U	1, 2, 5, 13, 32, 76, 176, 400, 896, 1984	A098156
D	1, 0, 2, 3, 7, 15, 33, 72, 157, 341	
Peak	1, 1, 3, 7, 16, 36, 80, 176, 384, 832	A045891
Ret	1, 1, 3, 6, 13, 27, 56, 115, 235, 478	A099036
Cat	0, 1, 1, 4, 8, 18, 38, 80, 166, 342	A175657
Δ_k	$\underbrace{0,\dots,0}_{k-1 \text{ zeroes}}$, 1, 0, 2, 3, 7, 15, 33, 72, 157, 341	
$\Delta_{\geqslant k}$	$\underbrace{0,\dots,0}_{k-1 \text{ zeroes}}$, 1, 1, 3, 6, 13, 28, 61, 133, 290, 631	New (v_n)
$\Delta_{\leqslant k}$	$\Delta_{\leqslant 1}$ 1, 0, 2, 3, 7, 15, 33, 72, 157, 341	$v_n - v_{n-k}$
	$\Delta_{\leqslant 2}$ 1, 1, 2, 5, 10, 22, 48, 105, 229, 498	
	$\Delta_{\leqslant 3}$ 1, 1, 3, 5, 12, 25, 55, 120, 262, 570	
	etc.	

Grand Dyck paths with air pockets, $\mathcal{G}_n$

A *grand Dyck path with air pockets* is a the same object as a regular Dyck path with air pockets, except it is allowed to dip below the x-axis. The set of such paths is denoted by $\mathcal{G}$.

Let $\mathcal{G}_n$ denote the set of n-length grand Dyck paths with air pockets.

Example:

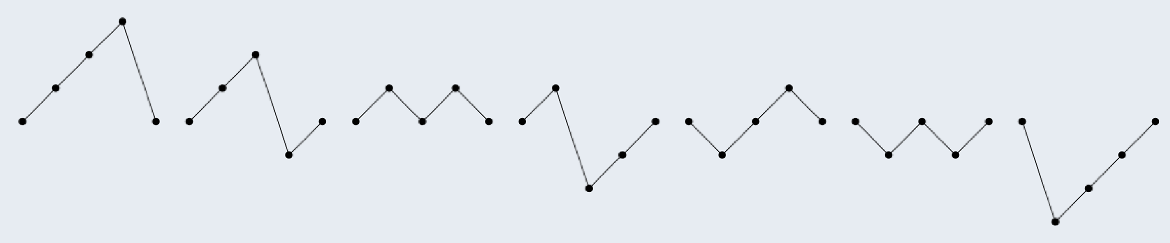


Figure: $\mathcal{G}_4 = \{U^3D_3, U^2D_3U, UDUD, UD_3U^2, DU^2D, DUDU, D_3U^3\}$.

Generating function (OEIS A051291):

$$\sum_{n=0}^{\infty} |\mathcal{G}_n| \cdot x^n = \frac{1 + 2x^2 - x^3 - (x-1)\sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2\sqrt{1 - 2x - x^2 - 2x^3 + x^4}}.$$

Taylor expansion:

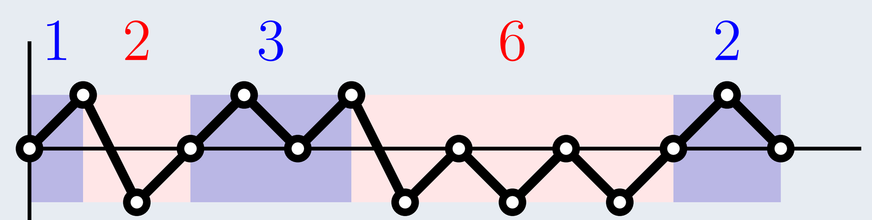
$$1 + 2x^2 + 3x^3 + 7x^4 + 17x^5 + 40x^6 + 97x^7 + 238x^8 + 587x^9 + 1458x^{10} + \dots$$

Bijection with certain compositions

The set $\mathcal{G}_n^{[-1,1]}$ of n-length grand Dyck paths with airpockets bounded by $y = \pm 1$ is in bijection with the set $\mathcal{C}(n+3)$ of compositions of $n+3$ such that the first part is odd, the last part is even, and no two consecutive parts have the same parity (OEIS A122514).

One bijection is obtained by decomposing the elements of $\mathcal{G}_n^{[-1,1]}$ into blocks, based on the occurrences of the patterns U^2 and UD_2, and then slightly modifying the list of these blocks' lengths.

Example:



The blocks of the path $UD_2U^2DUD_2UDUDU^2D \in \mathcal{G}_{14}^{[-1,1]}$ have successive lengths `1,2,3,6,2`, which we then reverse to obtain `2,6,3,2,1`, and finally we add 1 to the first entry and we append `2` to the end of the list (these final operations depend on the list we have, but in essence, we have to add 3 so that the sum of the entries is equal to $n+3$ in the end, and we do it in a manner that ensures the composition we obtain fits the definition of $\mathcal{C}(n+3)$). In this example, we get the composition

$$\underset{(14+3)}{17} = 3 + 6 + 3 + 2 + 1 + 2,$$

which is an element of $\mathcal{C}(17)$.

$\mathcal{G}_n^{[0,2]}$ (which is also a subset of $\mathcal{A}_n$) is in bijection with the set of compositions of $n-2$ such that no two consecutive parts have the same parity (OEIS A062200). One bijection can be obtained in a similar manner as what we just described between $\mathcal{G}_n^{[-1,1]}$ and $\mathcal{C}(n+3)$.