Skew Dyck paths with air pockets

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Plan of the talk

- A note about air pockets

Lattice paths with air pockets

Air pockets: down-steps of the form $(1, -k)$ ($k > 0$), two of which cannot be consecutive.

Baril, Kirgizov, M., Vajnovszki: Dyck paths with air pockets (2023), Grand Dyck paths with air pockets (2024).
$\longrightarrow$ Enumeration results, bijections with various objects (peakless Motzkin paths, Fibonacci meanders, integer compositions, ...)

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What is a skew DAP?

Lattice path in the first quadrant of $\mathbb{Z}^2$

Starts and ends on the x-axis

Steps: $U = (1,1)$, $L = (-1,-1)$, and $D_k = (1,-k)$ (for $k > 0$)

Forbidden patterns: UL, LU, and D_iD_j (for all $i, j > 0$)

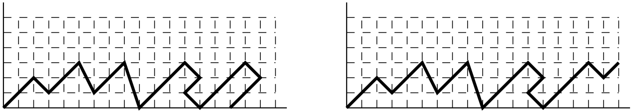

Figure 1: Left: skew DAP; Right: partial skew DAP ending at ordinate 3.

Enumeration of skew DAP

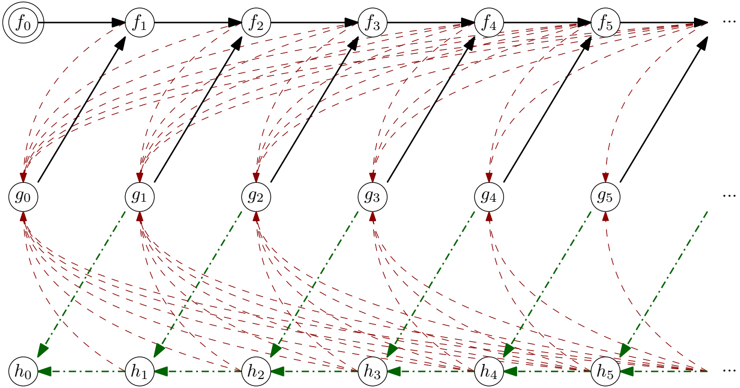

Figure 2: State graph of skew DAP

Black: U-step; Red: D_k-step; Green: L-step.

Enumeration of skew DAP

State graph translates into the following equations:

$$\begin{cases} & f_0(z) = 1, \\ \forall k > 0, & f_k(z) = z\left(f_{k-1}(z) + g_{k-1}(z)\right), \\ \forall k \geqslant 0, & g_k(z) = z\sum_{i\geqslant 1}\left(f_{k+i}(z) + h_{k+i}(z)\right), \\ \forall k \geqslant 0, & h_k(z) = z\left(g_{k+1}(z) + h_{k+1}(z)\right). \end{cases}$$

We set $F(z,u) = \sum_{k\geqslant 0} f_k(z)u^k$, $G(z,u) = \sum_{k\geqslant 0} g_k(z)u^k$, and $H(z,u) = \sum_{k\geqslant 0} h_k(z)u^k$.

$\longrightarrow$ Solve for $F(z,u) + G(z,u) + H(z,u)$, using the kernel method.
Kernel: $u^3z - 2u\,z^3 - u^2z + z^2u - u^2 + zu + z^2 + u - z$.

Enumeration of skew DAP

Result:

$$F(z,u)+G(z,u)+H(z,u)=\frac{s_1(1-z^2)}{z(s_1-z)(s_1-u)},$$

with

$$s_1=\frac{A}{6z}+\frac{4z^4-2z^3-\frac{4}{3}z^2-\frac{2}{3}z+\frac{2}{3}}{zA}+\frac{z+1}{3z},$$

$$A=\left(72z^5-72z^4+44z^3+12B\,z-48z^2-12z+8\right)^{\frac{1}{3}},$$

and

$$B=\big(-96z^{10}+144z^9+60z^8-108z^7-24z^6 \\ -48z^5+81z^4-18z^2+12z-3\big)^{1/2}.$$

Enumeration of skew DAP

Taylor expansion of $F(z,u) + G(z,u) + H(z,u)$:

$$
\begin{aligned}
1 + uz + (u^2+1)z^2 + (u^3+2u+1)z^3 + (u^4+3u^2+2u+3)z^4 \\
+ (u^5+4u^3+3u^2+6u+5)z^5 \\
+ (u^6+5u^4+4u^3+10u^2+11u+13)z^6 \\
+ (u^7+6u^5+5u^4+15u^3+19u^2+28u+26)z^7 + O(z^8).
\end{aligned}
$$

Resulting sequences when $u = 0$ (skew DAP) or $u = 1$ (partial skew DAP) are not in the OEIS.

$\underline{u = 0}$: $1, 0, 1, 1, 3, 5, 13, 26, 64, 143, \ldots$
$\underline{u = 1}$: $1, 1, 2, 4, 9, 19, 44, 100, 236, 558 \ldots$

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What is a valley-avoiding skew DAP?

Skew DAP, but any occurrence of $D_k U$ is forbidden (for all $k > 0$).

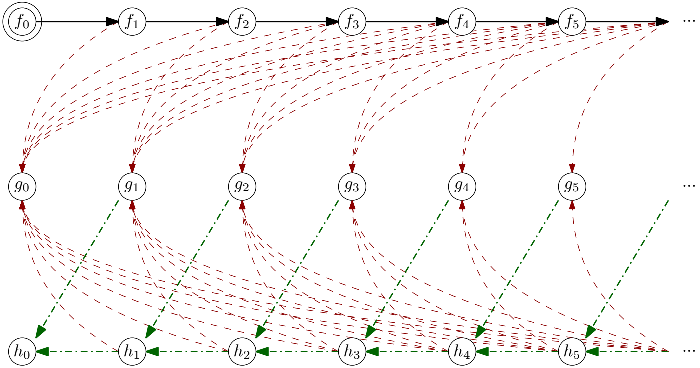

Figure 3: State graph of v.a. skew DAP

Black: U-step; Red: D_k-step; Green: L-step.

Enumeration of v.a. skew DAP

State graph translates into the following equations:

$$\begin{cases} & f_0(z) = 1, \\ \forall k > 0, & f_k(z) = z f_{k-1}(z), \\ \forall k \geqslant 0, & g_k(z) = z \sum_{i \geqslant 1} \left(f_{k+i}(z) + h_{k+i}(z)\right), \\ \forall k \geqslant 0, & h_k(z) = z \left(g_{k+1}(z) + h_{k+1}(z)\right). \end{cases}$$

We set $F(z,u) = \sum_{k \geqslant 0} f_k(z) u^k$, $G(z,u) = \sum_{k \geqslant 0} g_k(z) u^k$, and $H(z,u) = \sum_{k \geqslant 0} h_k(z) u^k$.

$\longrightarrow$ Solve for $F(z,u) + G(z,u) + H(z,u)$, using the kernel method.
Kernel: $u^3 z^2 - u^2 z^3 - u\, z^4 - u^3 z + 2u\, z^3 + z^3 + u^2 - 2z^2 - u + z$.

Enumeration of v.a. skew DAP

Result:

$$F(z,u)+G(z,u)+H(z,u)=\frac{1-z+z^3-z^4}{(1-zu)(1-z-z^2+z^3-z^4)}.$$

Taylor expansion:

$$\begin{aligned}1&+uz+\left(u^2+1\right)z^2+\left(u^3+u+1\right)z^3+\left(u^4+u^2+u+2\right)z^4\\&+\left(u^5+u^3+u^2+2u+2\right)z^5\\&+\left(u^6+u^4+u^3+2u^2+2u+4\right)z^6\\&+\left(u^7+u^5+u^4+2u^3+2u^2+4u+5\right)z^7+\mathrm{O}\left(z^8\right).\end{aligned}$$

$\underline{u=0}$: $1,0,1,1,2,2,4,5,9,12,\dots$ (OEIS A124280)
$\underline{u=1}$: $1,1,2,3,5,7,11,16,25,37,\dots$ (OEIS A130137)

Bijection with a set of binary words

$F(z,1) + G(z,1) + H(z,1)$ is the same g.f. as that of binary words avoiding both 00 and 0110.

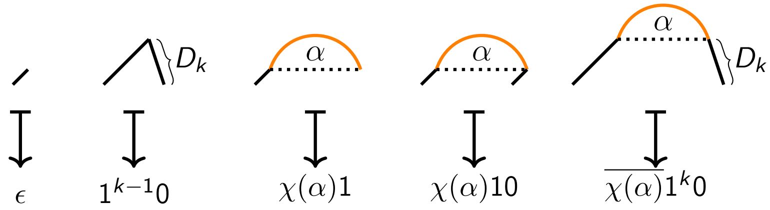

Figure 4: Illustration of the bijection χ.

$\overline{w_1 w_2 \dots w_{n-1} w_n} := w_1 w_2 \dots w_{n-1} 1$ for any word $w_1 w_2 \dots w_{n-1} w_n$.

Example: $\chi(U^4 DLD_2) = 011110$.

Bijection with a set of binary words

n	$\alpha \in \mathcal{PV}_n$	$\chi(\alpha) \in \mathcal{B}_{n-1}^{(00,0110)}$
1	U	ε
2	UD	0
	UU	1
3	UUD	01
	UUD_2	10
	UUU	11
4	$UUDL$	010
	$UUUD$	011
	$UUUD_2$	101
	$UUUD_3$	110
	$UUUU$	111

Table 1: First values of χ.

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What is a zigzagging, valley-avoiding skew DAP?

V.a. skew DAP, but any occurrence of LL is forbidden.

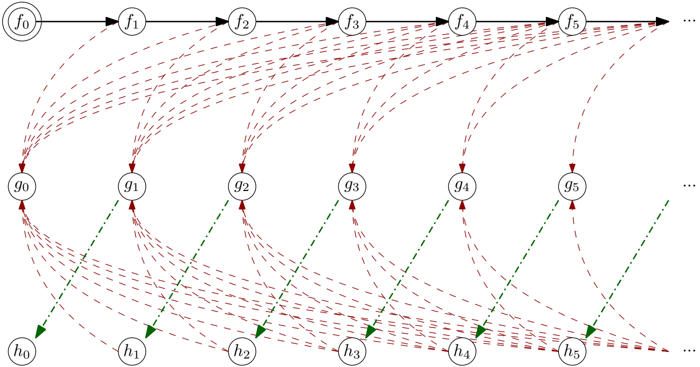


Figure 5: State graph of v.a. skew DAP

Black: U-step; Red: D_k-step; Green: L-step.

Enumeration of z.v.a. skew DAP

State graph translates into the following equations:

$$\begin{cases} & f_0(z) = 1, \\ \forall k > 0, & f_k(z) = zf_{k-1}(z), \\ \forall k \geqslant 0, & g_k(z) = z\sum_{i\geqslant 1}\left(f_{k+i}(z) + h_{k+i}(z)\right), \\ \forall k \geqslant 0, & h_k(z) = zg_{k+1}(z). \end{cases}$$

We set $F(z,u) = \sum_{k\geqslant 0} f_k(z)u^k$, $G(z,u) = \sum_{k\geqslant 0} g_k(z)u^k$, and $H(z,u) = \sum_{k\geqslant 0} h_k(z)u^k$.

$\longrightarrow$ Solve for $F(z,u) + G(z,u) + H(z,u)$, using the kernel method.
Kernel: $u^3z^2 - uz^4 - u^3z - u^2z^2 + uz^3 + z^3 + u^2 + uz - z^2 - u$.

Enumeration of z.v.a. skew DAP

Result:

$$F(z,u)+G(z,u)+H(z,u)=\frac{1-z+z^2}{(1-zu)(1-z-z^4)}.$$

Taylor expansion:

$$\begin{aligned}1 &+ uz + \left(u^2+1\right)z^2 + \left(u^3+u+1\right)z^3 + \left(u^4+u^2+u+2\right)z^4\\ &+ \left(u^5+u^3+u^2+2u+2\right)z^5\\ &+ \left(u^6+u^4+u^3+2u^2+2u+3\right)z^6\\ &+ \left(u^7+u^5+u^4+2u^3+2u^2+3u+4\right)z^7 + \mathrm{O}\left(z^8\right).\end{aligned}$$

$\underline{u=0}$: $1,0,1,1,2,2,3,4,6,8,\dots$ (OEIS A103632)
$\underline{u=1}$: $1,1,2,3,5,7,10,14,20,28,\dots$ (not in the OEIS)

Bijection with a set of compositions

$F(z,0) + G(z,0) + H(z,0)$ is the same g.f. as that of palindromic compositions with parts in $\{2, 1, 3, 5, 7, 9, \ldots\}$.
$\psi(UD) = \varepsilon$, $\psi(U^2D_2) = (1)$, $\psi(U^2DL) = (2)$, $\psi(U^3D_2L) = (3)$.

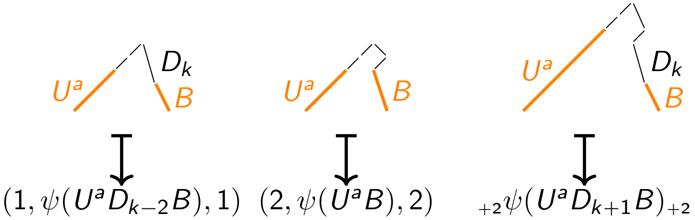


Figure 6: Illustration of the bijection ψ (nontrivial cases).

$$_{+2}(x_1)_{+2} := (x_1 + 4)$$

$$_{+2}(x_1, x_2, \ldots, x_{n-1}, x_n)_{+2} := \Big((x_1 + 2), x_2, \ldots, x_{n-1}, (x_n + 2)\Big)$$

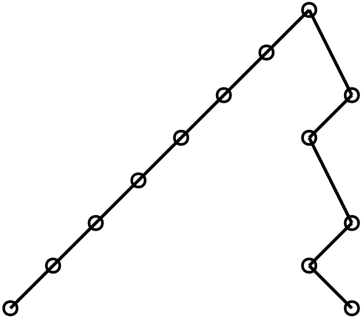
Bijection with a set of compositions

n	$\alpha \in \mathcal{Z}_n$	$\psi(\alpha) \in \mathcal{C}_{n-2}$
2	UD	ϵ
3	UUD_2	(1)
4	$UUUD_3$	$(1,1)$
	$UUDL$	(2)
5	$UUUUD_4$	$(1,1,1)$
	$UUUD_2L$	(3)
6	$UUUUUD_5$	$(1,1,1,1)$
	$UUUUD_3L$	$(1,2,1)$
	$UUUDLD$	$(2,2)$
7	$UUUUUUD_6$	$(1,1,1,1,1)$
	$UUUUUD_4L$	$(1,3,1)$
	$UUUUDLD_2$	$(2,1,2)$
	$UUUD_2LD$	(5)

Table 2: First values of ψ.

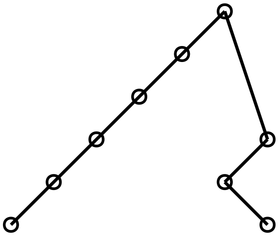
Example transformation

$\alpha = U^7 D_2 L D_2 L D =$



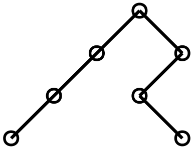
What is $\psi(U^7 D_2 L D_2 L D)$?

Example transformation



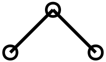
$$\psi(U^7 D_2 L D_2 L D) = {}_{+2}\psi(U^5 D_3 L D)_{+2}$$

Example transformation



$$\psi(U^7D_2LD_2LD) = {}_{+2}\psi(U^5D_3LD)_{+2} = {}_{+2}(1, \psi(U^3DLD), 1)_{+2}$$

Example transformation



$$\psi(U^7 D_2 L D_2 L D) = {}_{+2}\psi(U^5 D_3 L D)_{+2} = {}_{+2}(1, \psi(U^3 D L D), 1)_{+2}$$
$$= {}_{+2}(1, 2, \psi(UD), 2, 1)_{+2} = (3, 2, 2, 3).$$

Enumeration summary

Type of paths	First terms	OEIS
Skew DAPs	$1, 0, 1, 1, 3, 5, 13, 26, 64, 143, \ldots$	New
Partial skew DAPs	$1, 1, 2, 4, 9, 19, 44, 100, 236, 558 \ldots$	New
v.a. skew DAPs	$1, 0, 1, 1, 2, 2, 4, 5, 9, 12, \ldots$	A124280
Partial v.a. skew DAPs	$1, 1, 2, 3, 5, 7, 11, 16, 25, 37, \ldots$	A130137
z.v.a skew DAPs	$1, 0, 1, 1, 2, 2, 3, 4, 6, 8, \ldots$	A103632
Partial z.v.a skew DAPs	$1, 1, 2, 3, 5, 7, 10, 14, 20, 28, \ldots$	New

Thank you :)