Introduction to Dyck Paths with Air Pockets

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Plan of the talk

1. Dyck paths with air pockets (DAP)
   - Enumeration
   - Patterns
   - Subsets
2. Grand Dyck paths with air pockets (GDAP)
   - Enumeration
   - Subsets

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What are DAP?

Essentially like classical Dyck paths, except all runs of down-steps are condensed into singular, abrupt drops. The set of n-length DAP is $\mathcal{A}_n$.

Examples:

- UD is the only 2-length DAP;
- $\mathcal{A}_4 = \{UDUD, UUUD_3\}$;
- The empty path ε is a 0-length DAP.

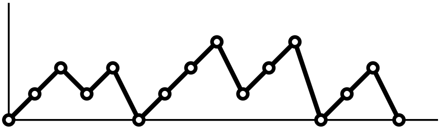


Figure 1: The DAP $UUDUD_2UUUD_2UUD_3UUD_2 \in \mathcal{A}_{15}$.

Goal and method

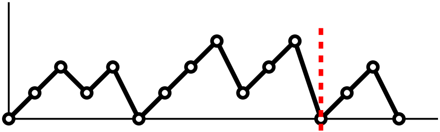
For each n, how many n-length DAP are there?
$\longrightarrow$ Find a closed formula (generating function) for the series

$$A(x) := \sum_{\sigma \in \mathcal{A}} x^{|\sigma|} = \sum_{n} |\mathcal{A}_n| \cdot x^n$$

by using a recursive description of DAP.

Decomposition of DAP

Second-to-last return decomposition:



$$\mathcal{A} = \varepsilon + \mathcal{A} \cdot UD + \mathcal{A} \cdot U\,(\mathcal{A} \setminus \{\varepsilon\})$$
$$A(x) = 1 + x^2 \cdot A(x) + x \cdot A(x) \cdot (A(x) - 1)$$
$$A(x) = \frac{1-x-x^2-\sqrt{x^4-2\,x^3-x^2-2\,x+1}}{2x}$$

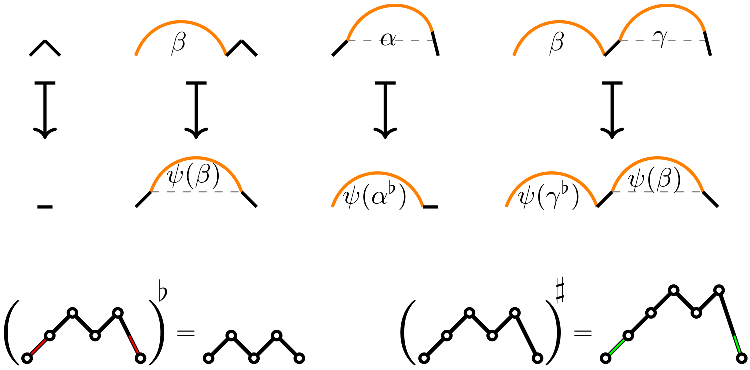
We have

$$A(x) = 1+x^2+x^3+2x^4+4x^5+8x^6+17x^7+37x^8+82x^9+185x^{10}+\ldots$$

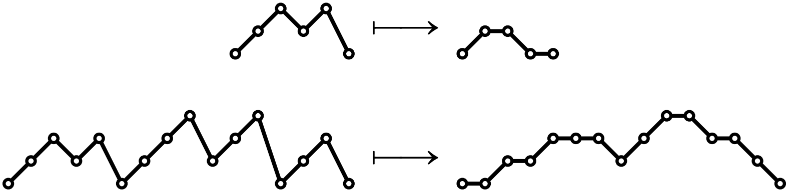
$\longrightarrow$ Generalized Catalan nbrs $(1, 0, 1, 1, 2, 4, 8, 17, 37, 82, 185, \ldots)$

Bijection with peakless Motzkin paths

Same cardinality as $(n-1)$-length peakless Motzkin paths (i.e. UD-avoiding Motzkin paths).
Bijection:



Examples



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Pattern distributions: goal and method

For each n, how often does a certain pattern p (say UUD_3 for example) show up in the set $\mathcal{A}_n$?
$\longrightarrow$ Find the generating function

$$A(x,y) := \sum_{\sigma \in \mathcal{A}} x^{|\sigma|} y^{\#_{\mathtt{p}}(\sigma)} = \sum_{n,k} a_{n,k} \cdot x^n \cdot y^k,$$

where $a_{n,k}$ is the number of elements of $\mathcal{A}_n$ where the pattern p appears exactly k times, by using a recursive description of DAP (e.g. the one we used previously).

Example

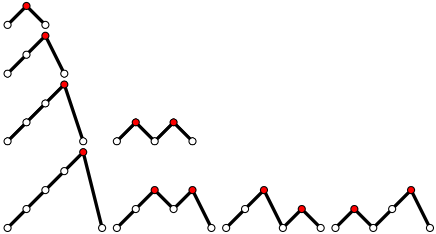
$\sum_m UD_m$ (peaks). Second-to-last return decomposition gives

$$A(x,y) = (1 + A(x,y)) \cdot \left(x^2 y + xA(x,y)\right)$$

$$A(x,y) = \frac{1-x-x^2y-\sqrt{(1-x-x^2y)^2-4x^3y}}{2x}$$

We have

$$\begin{aligned} A(x,y) = 1 & \\ &+ x^2y \\ &+ x^3y \\ &+ x^4y + x^4y^2 \\ &+ x^5y + 3x^5y^2 \\ &+ x^6y + 6x^6y^2 + x^6y^3 \\ &+ \ldots \end{aligned}$$



Pattern popularities

For each n, what is the total number of times the pattern p appears in $\mathcal{A}_n$?

$\longrightarrow$ Closed formula for $\sum_n \left(\sum_k k \cdot a_{n,k}\right) \cdot x^n$.

$\longrightarrow$ Derived from distribution $A(x, y)$: compute $\frac{\partial}{\partial y} A(x, y)|_{y=1}$.

Example #1

The popularity function of peaks ($\sum_m UD_m$) is

$$\frac{\partial}{\partial y} A(x,y)|_{y=1} = \frac{x\left(1+x-x^2-\sqrt{(1-x-x^2)^2-4x^3}\right)}{2\sqrt{(1-x-x^2)^2-4x^3}}$$

Taylor coefficients: $0, 0, 1, 1, 3, 7, 16, 39, 95, 233, \ldots$

$\longrightarrow$ n-th coefficient $\sim \frac{\sqrt{5}-2}{\sqrt{\pi n}\sqrt{14\sqrt{5}-30}} \left(\frac{3+\sqrt{5}}{2}\right)^n$ as $n \to \infty$.

$\longrightarrow$ Average number of peaks in a typical n-length DAP:

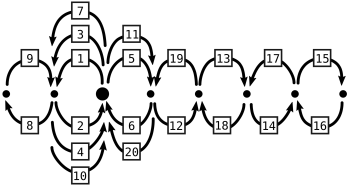
$$\sim \frac{5-\sqrt{5}}{10} n \approx 0.276393191 \cdot n \text{ as } n \to \infty.$$

Example #2

The popularity function of pyramids of size at least k ($\sum_{m\geqslant k} U^m D_m$) is

$$F_k(x) := \frac{x^{k+1}\left(1+2x^2-x^3+(1-x)\sqrt{x^4-2x^3-x^2-2x+1}\right)}{2(1-x)\sqrt{x^4-2x^3-x^2-2x+1}}.$$

$\longrightarrow$ Link with even-length Fibonacci meanders of modulus 2 (introduced by Peter Luschny): if G is the g.f. of such meanders, then $G(x) = \frac{F_k(x)}{x^{k+1}} - 1$ for all k.



Number of $2n$-length Fibonacci meanders of modulus 2 ($n \geqslant 1$):

$1, 3, 6, 13, 30, 70,$ (OEIS A201631)

$167, 405, 992, 2450, \ldots$

List of known sequences

Pattern	Pattern popularity in $\mathcal{A}_n$	OEIS
U	$1, 2, 5, 13, 32, 80, 201, 505, 1273, 3217$	A110320
D	$1, 0, 2, 3, 7, 17, 40, 97, 238, 587$	A051291
Peak	$1, 1, 3, 7, 16, 39, 95, 233, 577, 1436$	A203611
Ret	$1, 1, 3, 6, 13, 29, 65, 148, 341, 793$	A093128
Cat	$0, 1, 1, 4, 8, 19, 44, 102, 239, 563$	
Δ_k	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 0, 2, 3, 7, 17, 40, 97, 238, 587$	A051291
$\Delta_{\geqslant k}$	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 1, 3, 6, 13, 30, 70, 167, 405$	A201631$(= u_n)$
$\Delta_{\leqslant k}$	$\Delta_{\leqslant 1}$ $1, 0, 2, 3, 7, 17, 40, 97, 238, 587$	$u_n - u_{n-k}$
	$\Delta_{\leqslant 2}$ $1, 1, 2, 5, 10, 24, 47, 137, 335, 825, \ldots$	
	$\Delta_{\leqslant 3}$ $1, 1, 3, 5, 12, 27, 64, 154, 375, 922, \ldots$	
	$\vdots$	

Table 1: Pattern popularity in $\mathcal{A}_n$, for $2 \leqslant n \leqslant 11$.

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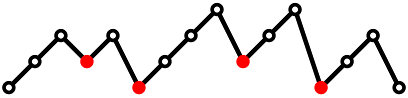
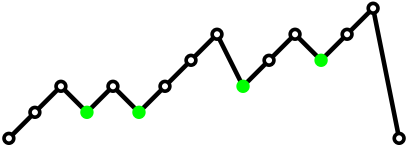
Non-decreasing DAP

DAP whose valleys are non-decreasing in altitude. Example:



is *not* non-decreasing, but



is.

List of known sequences

Pattern	Pattern popularity in $\mathcal{A}_n^{\nearrow}$	OEIS
U	$1, 2, 5, 13, 32, 76, 176, 400, 896, 1984$	A098156
D	$1, 0, 2, 3, 7, 15, 33, 72, 157, 341$	
Peak	$1, 1, 3, 7, 16, 36, 80, 176, 384, 832$	A045891
Ret	$1, 1, 3, 6, 13, 27, 56, 115, 235, 478$	A099036
Cat	$0, 1, 1, 4, 8, 18, 38, 80, 166, 342$	A175657
Δ_k	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 0, 2, 3, 7, 15, 33, 72, 157, 341$	
$\Delta_{\geqslant k}$	$\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 1, 3, 6, 13, 28, 61, 133, 290, 631$	New ($= v_n$)
$\Delta_{\leqslant k}$	$\Delta_{\leqslant 1}$ $1, 0, 2, 3, 7, 15, 33, 72, 157, 341$	$v_n - v_{n-k}$
	$\Delta_{\leqslant 2}$ $1, 1, 2, 5, 10, 22, 48, 105, 229, 498$	
	$\Delta_{\leqslant 3}$ $1, 1, 3, 5, 12, 25, 55, 120, 262, 570$	
	$\vdots$	

Table 2: Pattern popularity in $\mathcal{A}_n^{\nearrow}$ for $2 \leqslant n \leqslant 11$.

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What are GDAP?

Essentially like DAP, except they are allowed to go below the x-axis.
The set of n-length GDAP is $\mathcal{G}_n$.
Examples:

- UD and DU are the two 2-length GDAP;
- $\mathcal{G}_4 = \{UDUD, UUUD_3, UUD_3U, UD_3UU, D_3UUU, DUDU, DUUD\}$;
- The empty path ε is a 0-length GDAP.

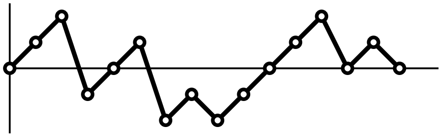

Figure 2: The GDAP $UUD_3UUD_3UDUUUUD_2UD \in \mathcal{G}_{15}$.

Decomposition of GDAP

A bit more involved, but it gives the following closed formula for the series $G(x) := \sum_{\sigma \in \mathcal{G}} x^{|\sigma|} = \sum_n |\mathcal{G}_n| \cdot x^n$:

$$G(x) = \frac{x^4 - 2x^3 - x^2 - 2x + 1 + (1 - x + x^2)R}{(1 + x - x^2 + R)R},$$

with $R = \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}$. We have

$$G(x) = 1 + 2x^2 + 3x^3 + 7x^4 + 17x^5 + 40x^6 + 97x^7 + 238x^8 + \dots$$

$\longrightarrow$ Same sequence as "Whitney number of level n of the lattice of the ideals of the fence of order $2n + 1$" (OEIS A051291).

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Minorized partial GDAP

I.e. prefixes of GDAP (they can end at any altitude, not just 0) staying above a certain ordinate.
Fairly involved computation method, but for example, the generating function of n-length partial GDAP above the altitude (depth) $m < 0$ is

$$\frac{F^{-m} - F^{-1-m} - x^2}{x^3},$$

with $F = \frac{1+x-x^2-\sqrt{x^4-2x^3-x^2-2x+1}}{2x}$.

$\underline{m=-1}$: $1, 2, 4, 8, 17, 37, 82, 185, \ldots$ (generalized Catalan numbers)
$\underline{m=-2}$: $1, 3, 6, 13, 29, 65, 148, \ldots$ (dissections of a polygon using strictly disjoint diagonals)

Bounded GDAP ($y \in [0, t]$)

More precisely, GDAP staying between altitudes 0 and $t > 0$.
Even more tedious computation: the generating function is

$$\frac{N_{t+1}^t}{D_t},$$

with

$$N_{t+1}^t = \frac{2^{t+2}x^{t+3}(-1)^t}{W(x^2-x-1)^2-W^3}\left(\frac{1}{(x^2-x-1+W)^t} - \frac{1}{(x^2-x-1-W)^t}\right),$$

$$D_t = \frac{2^t x^{t+1}}{W}\left(\frac{W-x^2+x-1}{(W-x^2+x+1)^{t+1}} + (-1)^{t+1}\frac{W+x^2-x+1}{(W+x^2-x-1)^{t+1}}\right),$$

and $W = \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}$.
$\longrightarrow$ If $t = 2$, n-length paths are in bijection with a set of compositions of the integer $n-2$ (OEIS A062200).

Matrix equation for bounded GDAP ($y \in [0, t]$)

$$\left[\begin{array}{cccc|cccc}
-1 & & & & 0 & & & \\
x & -1 & & & x & 0 & & \\
 & \ddots & \ddots & & & \ddots & \ddots & \\
 & & x & -1 & & & x & 0 \\
\hline
0 & x & \dots & x & -1 & & & \\
 & 0 & \ddots & \vdots & & \ddots & & \\
 & & \ddots & x & & & \ddots & \\
 & & & 0 & & & & -1
\end{array}\right] \cdot \left[\begin{array}{c} f_0^t \\ \vdots \\ \vdots \\ f_t^t \\ \hline g_0^t \\ \vdots \\ \vdots \\ g_t^t \end{array}\right] = \left[\begin{array}{c} -1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{array}\right]$$

Here, f_k^t (resp. g_k^t) is the g.f. of partial GDAP, bounded by $y = 0$ and $y = t$, ending at ordinate k with U (resp. D_m, $m \geqslant 1$).

Other bounded GDAP ($y \in [-t, t]$)

More precisely, GDAP staying between altitudes $-t$ and t.
Yet even more arduous computation: the generating function is

$$\frac{D_{t-1}}{D_{2t}} \cdot \left(D_t + N_{t+1}^t\right),$$

with

$$N_{t+1}^t = \frac{2^{t+2}x^{t+3}(-1)^t}{W(x^2-x-1)^2-W^3}\left(\frac{1}{(x^2-x-1+W)^t} - \frac{1}{(x^2-x-1-W)^t}\right),$$

$$D_t = \frac{2^t x^{t+1}}{W}\left(\frac{W-x^2+x-1}{(W-x^2+x+1)^{t+1}} + (-1)^{t+1}\frac{W+x^2-x+1}{(W+x^2-x-1)^{t+1}}\right),$$

and $W = \sqrt{x^4 - 2x^3 - x^2 - 2x + 1}$.
$\longrightarrow$ If $t = 1$, n-length paths are in bijection with an analogous set of compositions of the integer $n + 3$ (OEIS A122514).

Special case

If $t = 1$ and , the generating function is

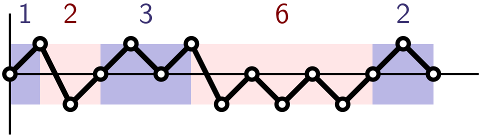
$$\frac{1}{x^4 - x^3 - 2x^2 + 1},$$

whose Taylor coefficients sequence is the same as the enumeration of certain integer compositions.
If n is an integer, consider the compositions of n where the parts alternate in parity, the first part is odd, and the last part is even (example: $14 = 3 + 4 + 5 + 2$). This set is $\mathcal{C}(n)$.

$\longrightarrow$ The result says the number of n-length GDAP bounded between altitudes -1 and 1 is exactly $|\mathcal{C}(n+3)|$ (OEIS A122514).

Bijection with $\mathcal{C}(n+3)$

Divide the bounded GDAP into blocks according to occurrences of UU and UD_2:

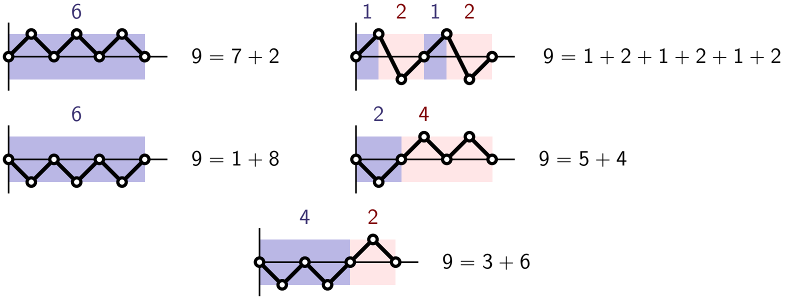


Reverse the list of the blocks' lengths, then fiddle with it so that it spells out an element of $\mathcal{C}(n+3)$ (there is only one way to do it*). Here: $2, 6, 3, 2, 1 \to 3, 6, 3, 2, 1, 2 \to 14 + 3 = 3 + 6 + 3 + 2 + 1 + 2$.

*unless the list is simply n, which either stems from $(UD)^{n/2}$ or $(DU)^{n/2}$, to which we associate $(n+1)+2$ and $1+(n+2)$, respectively.

Example: $n = 6$

Here are the five 6-length GDAP bounded by altitudes -1 and 1, and their corresponding element in $\mathcal{C}(9)$:



What's next?

Enumerate other subsets of $\mathcal{A}_n$ and $\mathcal{G}_n$.

Study pattern occurrences in GDAP.

(We are currently studying pattern avoidance in DAP.)

References:

- Baril-Kirgizov-Maréchal-Vajnovszki. *Enumeration of Dyck paths with air pockets* (2022), https://arxiv.org/abs/2202.06893
  $\longrightarrow$ (published in JIS, Vol. 26 (2023), Article 23.3.2)
- Baril-Kirgizov-Maréchal-Vajnovszki. *Grand Dyck paths with air pockets (2022)*, https://arxiv.org/abs/2211.04914

Thank you :)