LIPN nonpermanent seminar, 16/01/26

# The "phoenix" and "fork anywhere" models of git graphs

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# MOTIVATION

"DAGs are too broad; feature branch git graphs could be broadened."

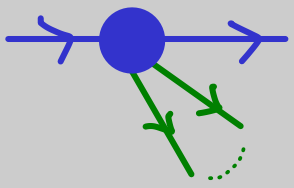
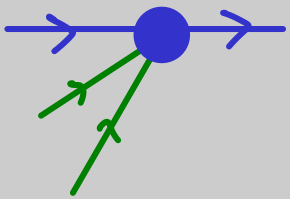
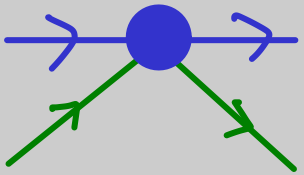
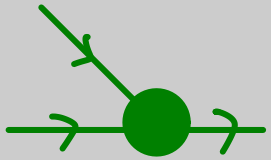
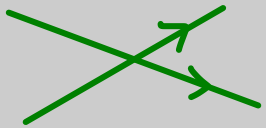
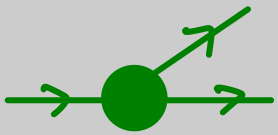
→ Come up with superclasses of feature branch git graphs, such that:

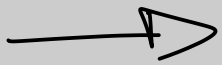
* they still describe realistic git workflows;
* their enumeration / random generation are not too difficult.

# STARTING POINT

Reminder

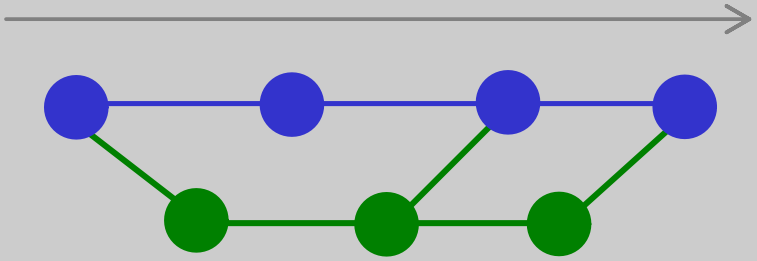
Feature branch graphs rules :

| OK | OK n't |
|---|---|
|  |  |
|  |  |
|  |  |



Broaden this class of graphs by allowing feature branches to be "reborn" when they go to "die" on the main branch,

e.g.



# PHOENIX GRAPHS

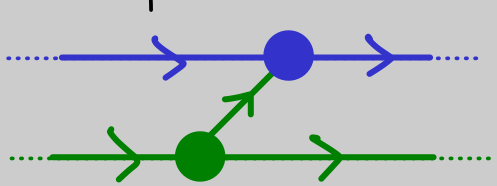
## Definition

Phoenix graph = DAG with:

- a linear main branch
- zero or more feature branches (DAGs starting on a main vertex, and ending on one or more feature vertices further to the right)

- indegree(●) $\in \{1,2\}$ ($=0$ for leftmost)
- outdegree(●) $\geq 1$ ($=0$ for rightmost)
- indegree(●) $= 1$
- outdegree(●) $\in \{1,2\}$, $=2$ iff one child of each colour

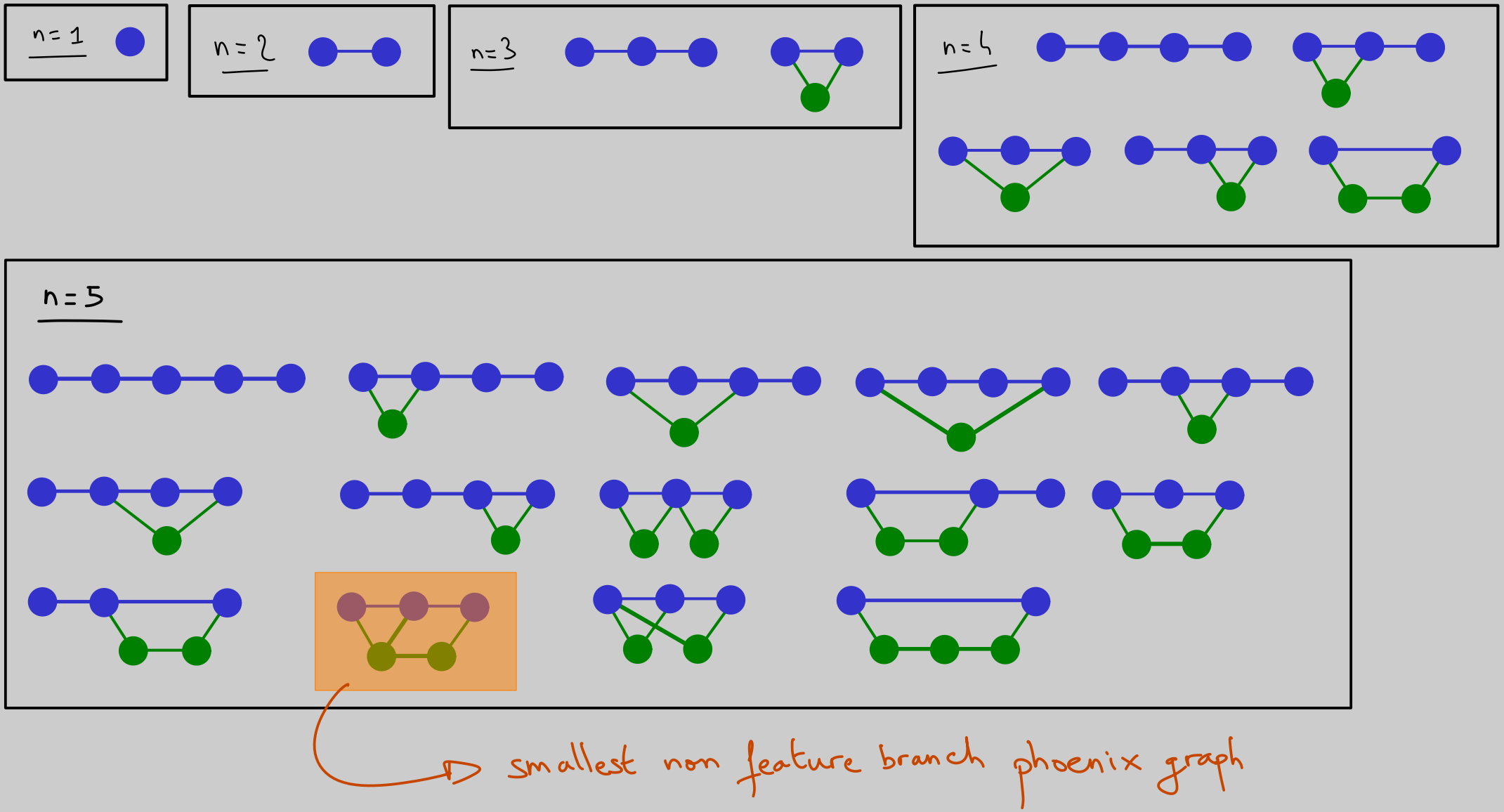
i.e a feature branch graph, but the pattern



is allowed.

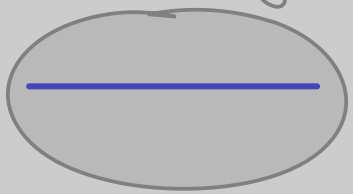
# EXAMPLES

Size $n$ of a phoenix graph = number of vertices

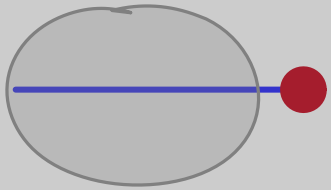


# RECURSIVE DECOMPOSITION

Phoenix graph

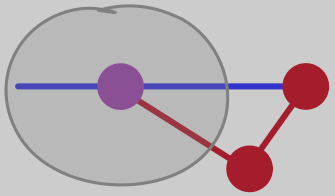


= [empty] or



or

k-1 choices



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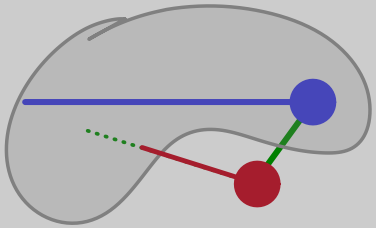
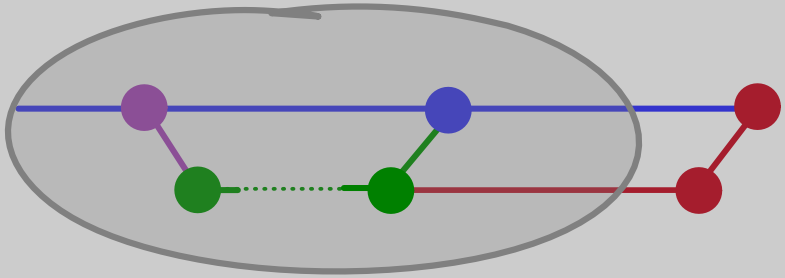


or

b choices



$$g_{n,k,b} = g_{n-1,k-1,b} + (k-1)\,g_{n-2,k-1,b-1} + (g_{n-1,k,b} - g_{n-2,k-1,b}) + b \cdot g_{n-2,k-1,b}$$

number of phoenix graphs with $n$ vertices, $k$ main vertices, $b$ feature branches

# GENERATING FUNCTION

★ $g_{n,k,b} = g_{n-1,k-1,b} + (k-1)\,g_{n-2,k-1,b-1} + g_{n-1,k,b} + (b-1)\,g_{n-2,k-1,b}$

★ $\widetilde{G}(z,u,v) := \sum_{n,k,b \geq 0} g_{n,k,b}\, z^n u^k v^b$

["$\widetilde{G}$" for short]

↳ not analytic...

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* $\widetilde{G}(z,u,v) := \sum_{n,k,b \geq 0} g_{n,k,b}\, z^n \frac{u^k}{k!} \frac{v^b}{b!}$ ["$\widetilde{G}$" for short]

$$\hookrightarrow z^2\widetilde{G} - z^2 u \partial_u \widetilde{G} - z\partial_v \widetilde{G} + (1-z)\partial_u \partial_v \widetilde{G} - z^2 v \partial_{v,v}^2 \widetilde{G} = 0$$

Sadly, no solution ☹

Moreover, the sequence $\left(\sum_{k,b} g_{n,k,b}\right)_{n \geq 0}$ does not appear in the OEIS.

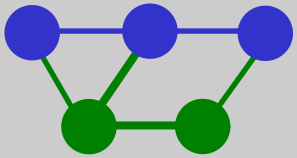
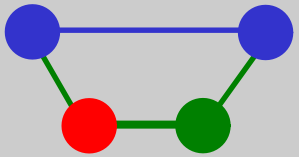
⟶ What else *can* we say about phoenix graphs?

# BABY PHOENIXES

## Definition

Baby phoenix graph = phoenix graph where every feature vertex is part of a merge (i.e one of its children is a main vertex, and no main vertex has its indegree equal to 1

## Example

 is a baby phoenix, but  is not.

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To construct a baby phoenix graph with $k \geq 1$ main vertices, start with the main branch:

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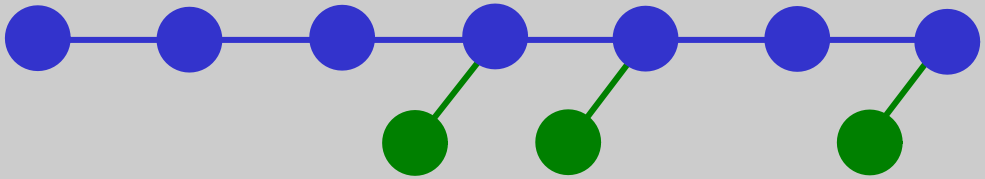
To construct a baby phoenix graph with $k \geq 1$ main vertices, start with the main branch:

Then, among the $k-1$ rightmost vertices, take a nonempty subset, make each one of them part of a merge (adding feature vertices as needed), make them part of the same feature branch, and finally choose where that branch starts:



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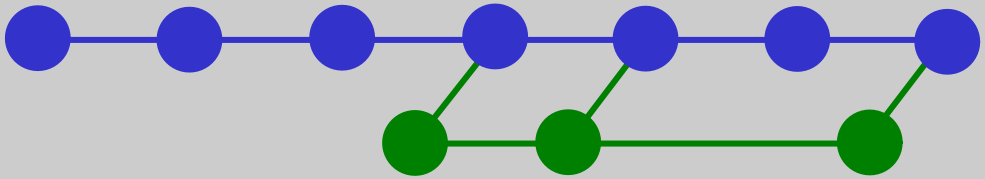

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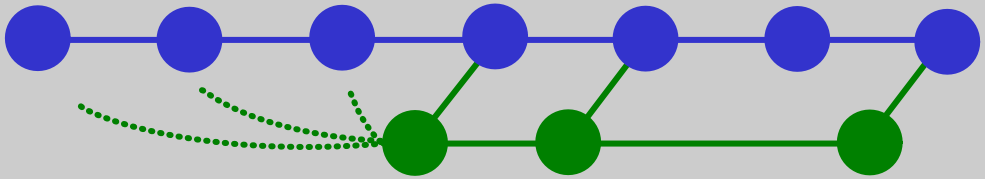

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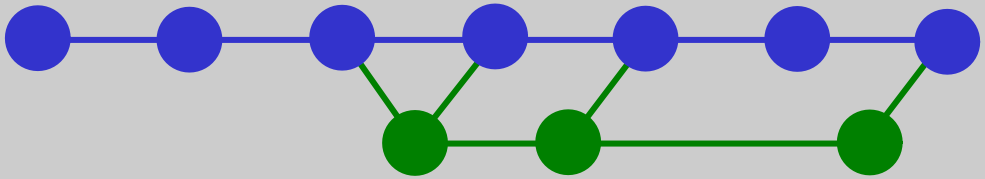

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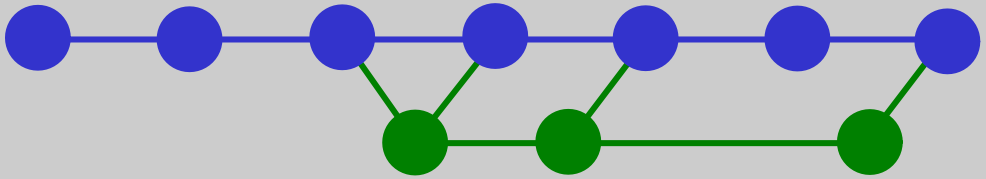
Repeat this process with the available main vertices to add other feature branches:



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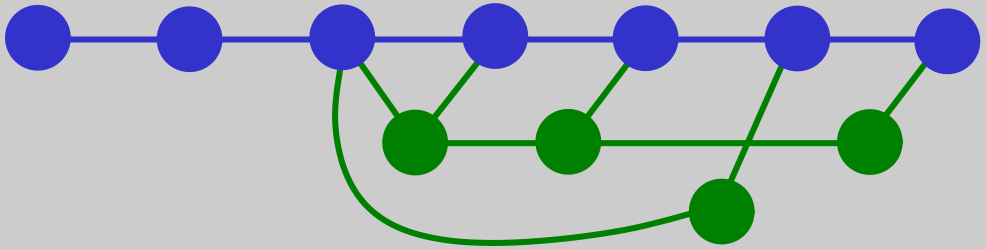

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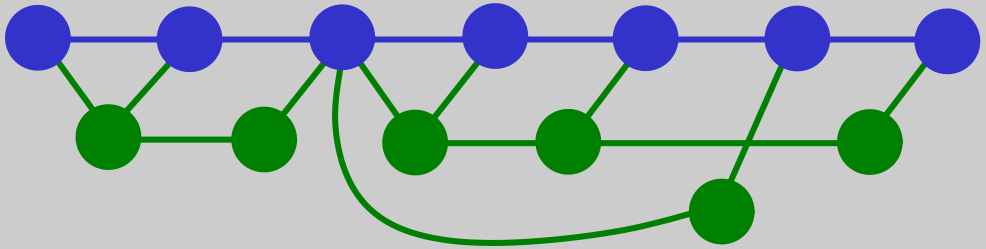

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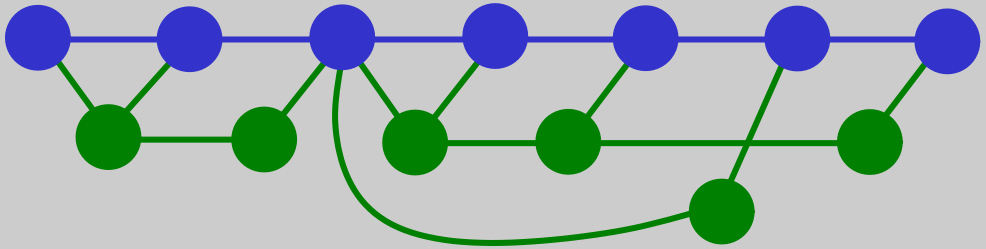

Notes

* A baby phoenix graph with $k \geq 1$ main vertices necessarily has $k-1$ feature vertices.
* Counting sequence: 1, 1, 3, 14, 89, 716, ...
  ↪ OEIS A007549

# BIJECTION WITH SET PARTITIONS

Labelling the main vertices from 0 to $k-1$, we can map a baby phoenix graph with $k$ main vertices to a set partition of size $k-1$:

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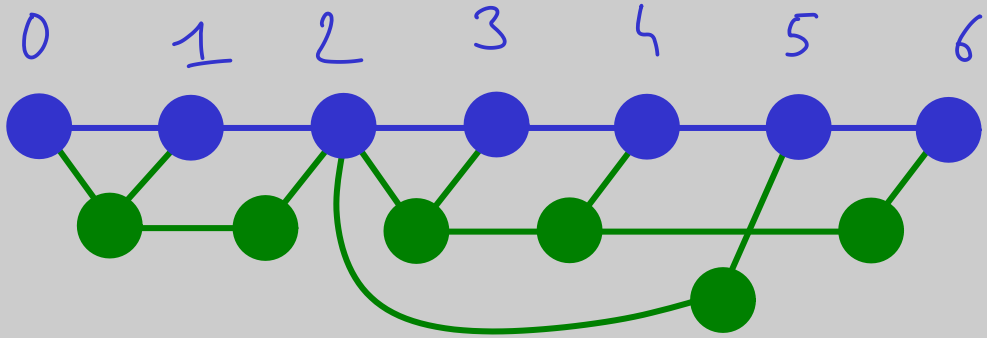

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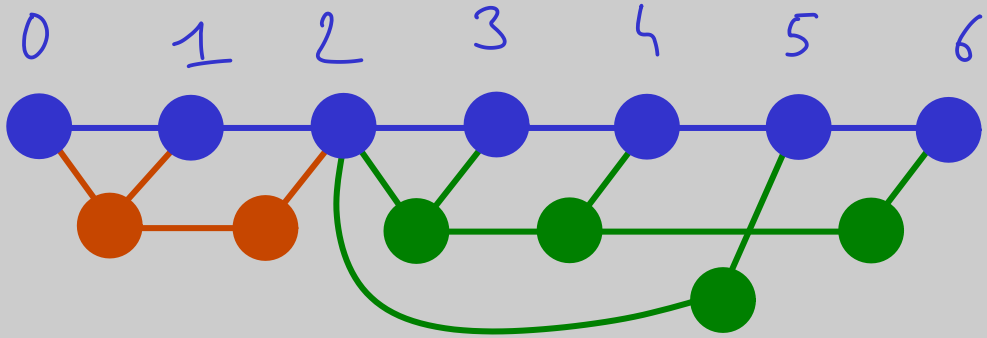


$\{1, 2\}$

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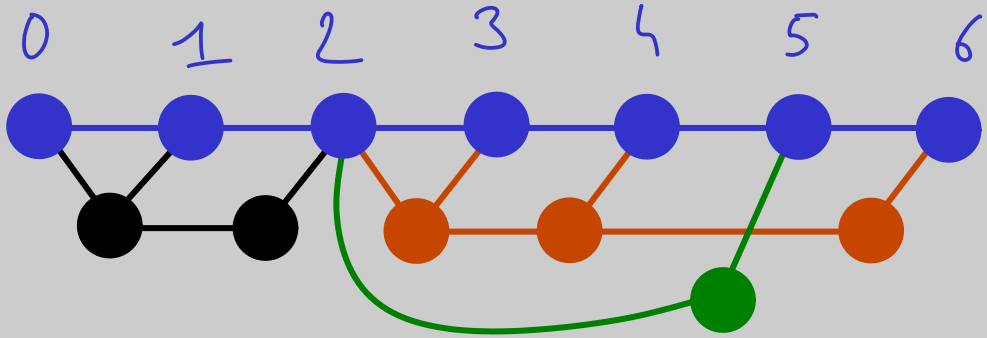


$\{1, 2\}$ $\{3, 4, 6\}$

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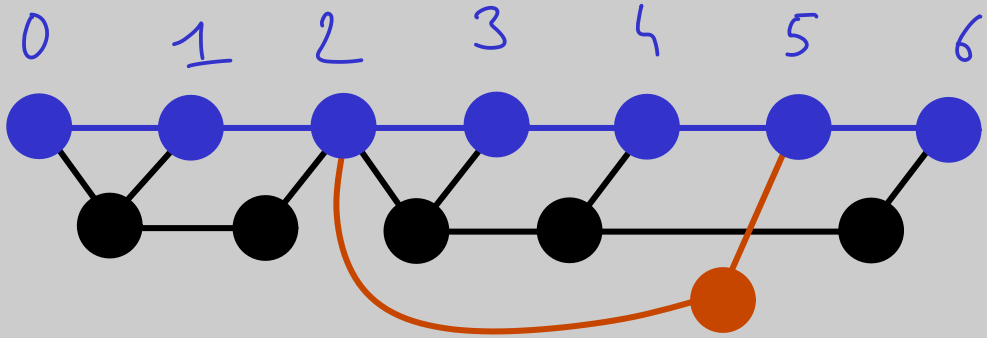


$\{1,2\}$ $\{3,4,6\}$ $\{5\}$

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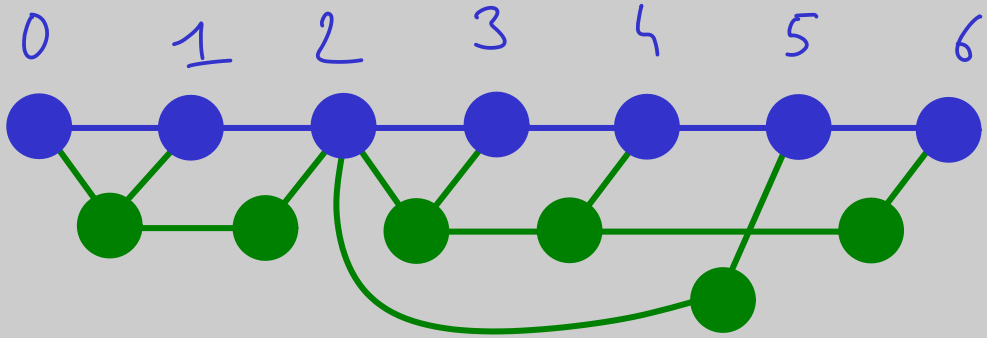


$$[6] = \{1,2\} \sqcup \{3,4,6\} \sqcup \{5\}$$

↳ This is a surjection. How do we make it a bijection?

# BIJECTION WITH SET PARTITIONS

How many ways are there to obtain a given part $\{k_1, k_2, \ldots, k_m\}$ (here, $k_1 < k_2 < \ldots < k_m$)?

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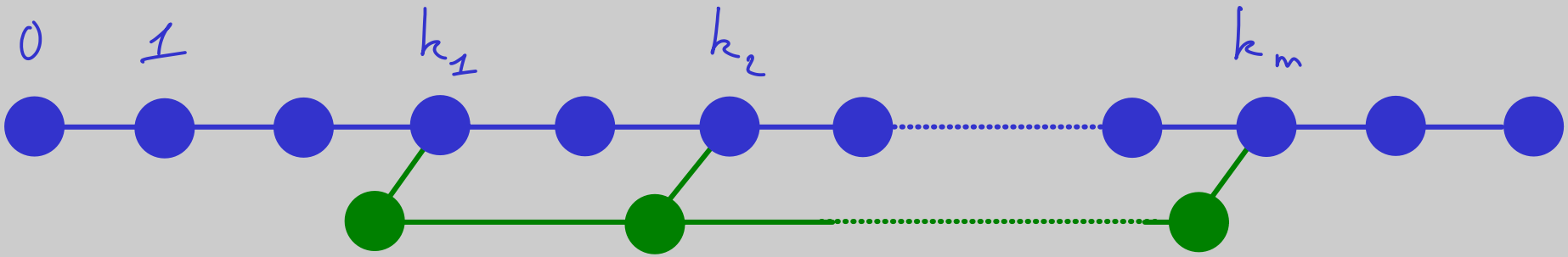

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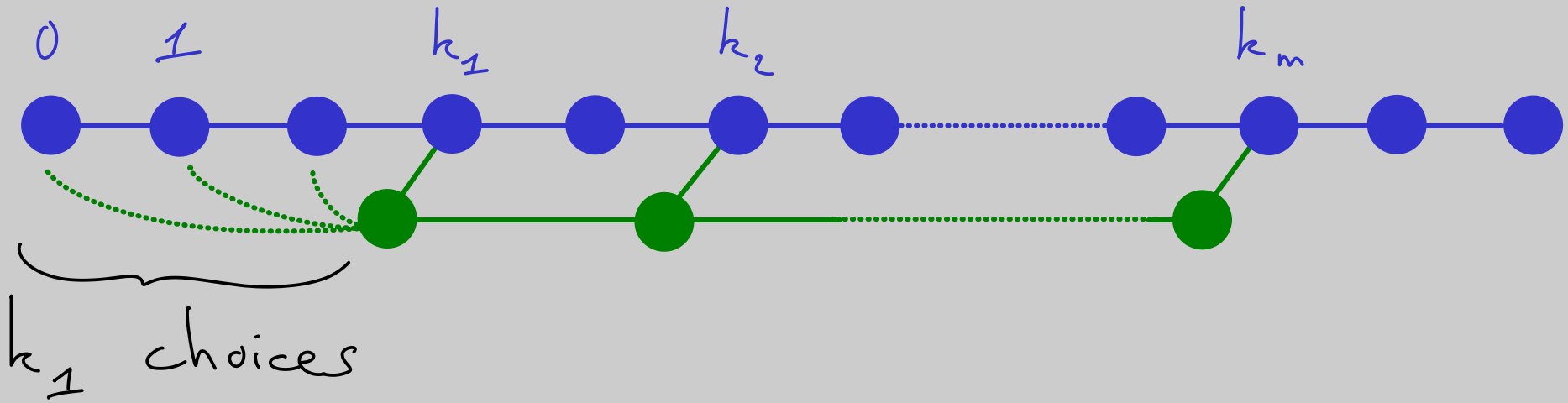

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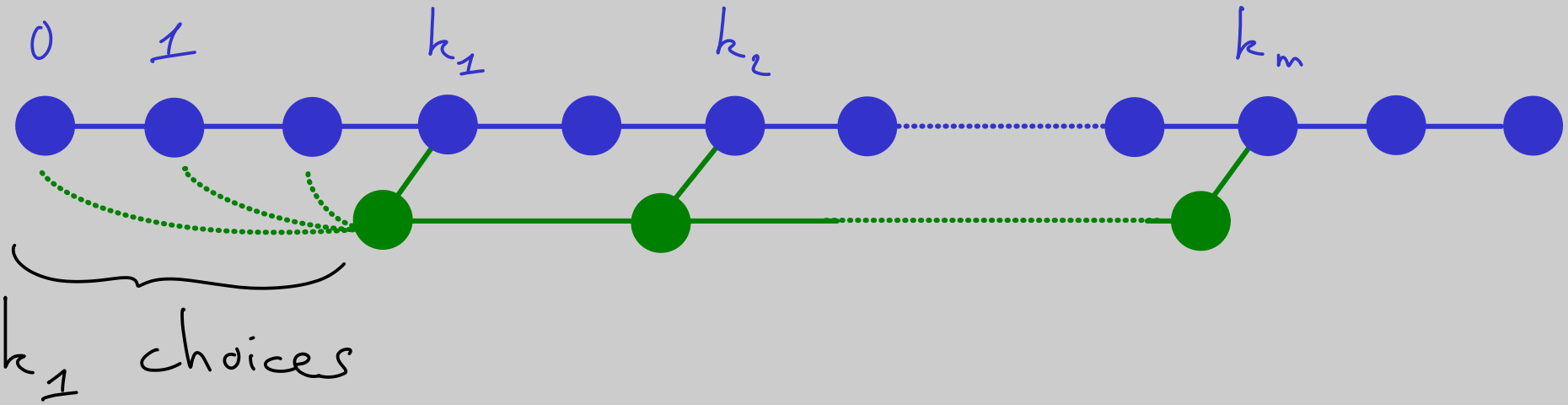



↳ Baby phoenix graphs with $k$ main vertices

$\simeq$

Set partitions of size $k-1$,

weighted by the product of part minima

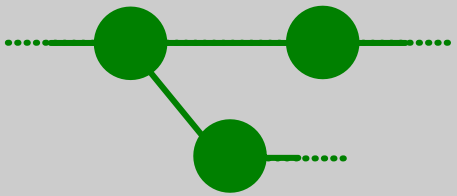
# FORK ANYWHERE

["FA" for short]

**Definition**

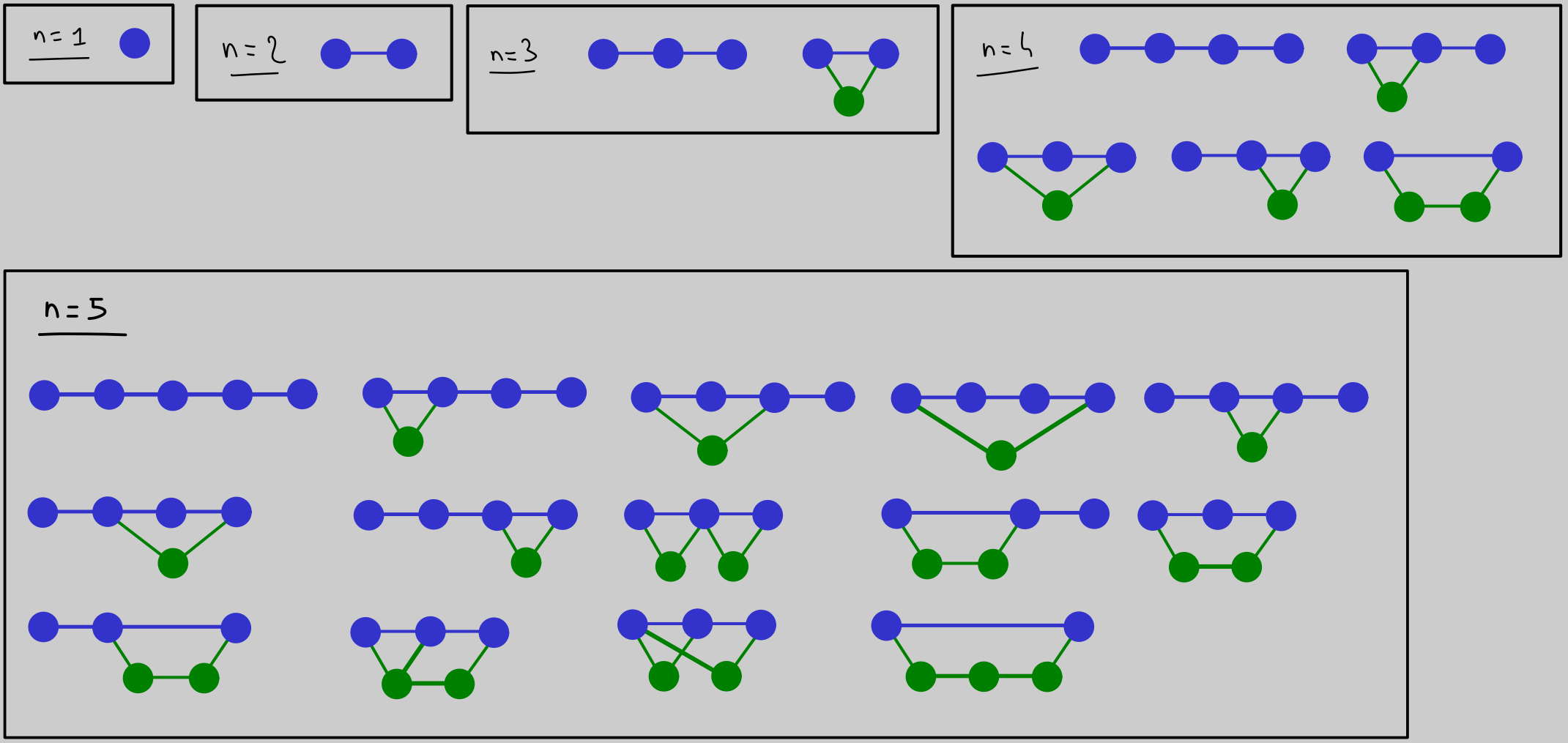
Fork anywhere graph = phoenix graph, but vertices on feature branches can have their outdegree equal to 2 without restriction on their children's types,

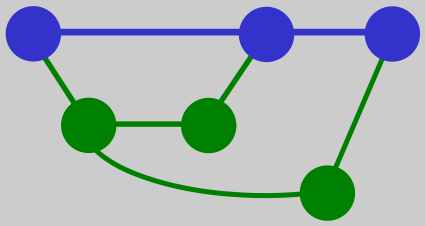
i.e. a phoenix graph, but the pattern



is allowed.

# EXAMPLES

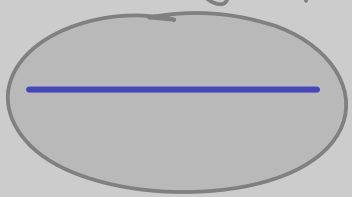


is the smallest non phoenix FA graph.

# RECURSIVE DECOMPOSITION

FA graph



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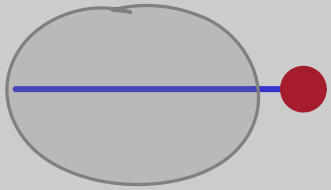

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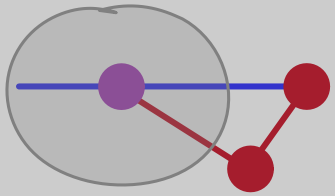

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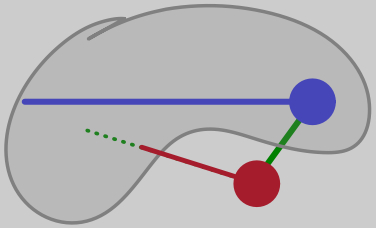

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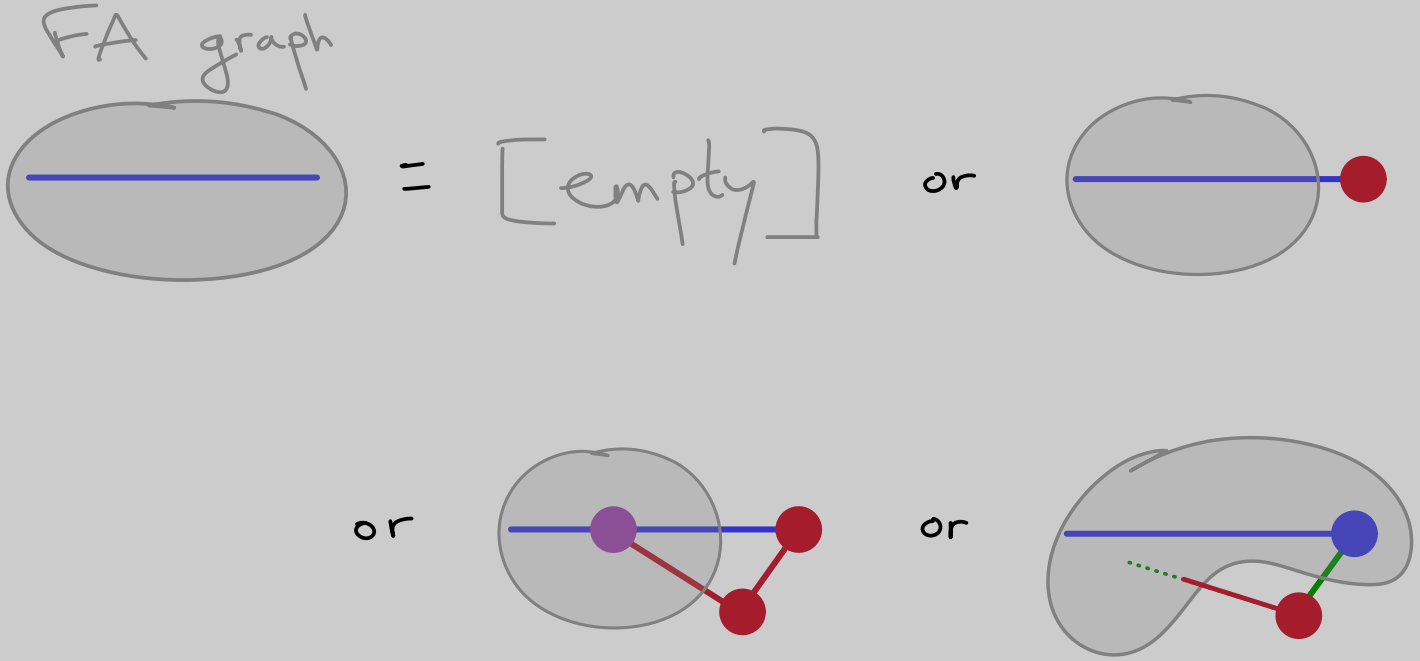


OGF "$G$":

$$G(z) = \sum_{n \geq 0} g_n z^n$$

EGF "$\tilde{G}$":

$$\tilde{G}(z) = \sum_{n \geq 0} \frac{g_n}{n!} z^n$$

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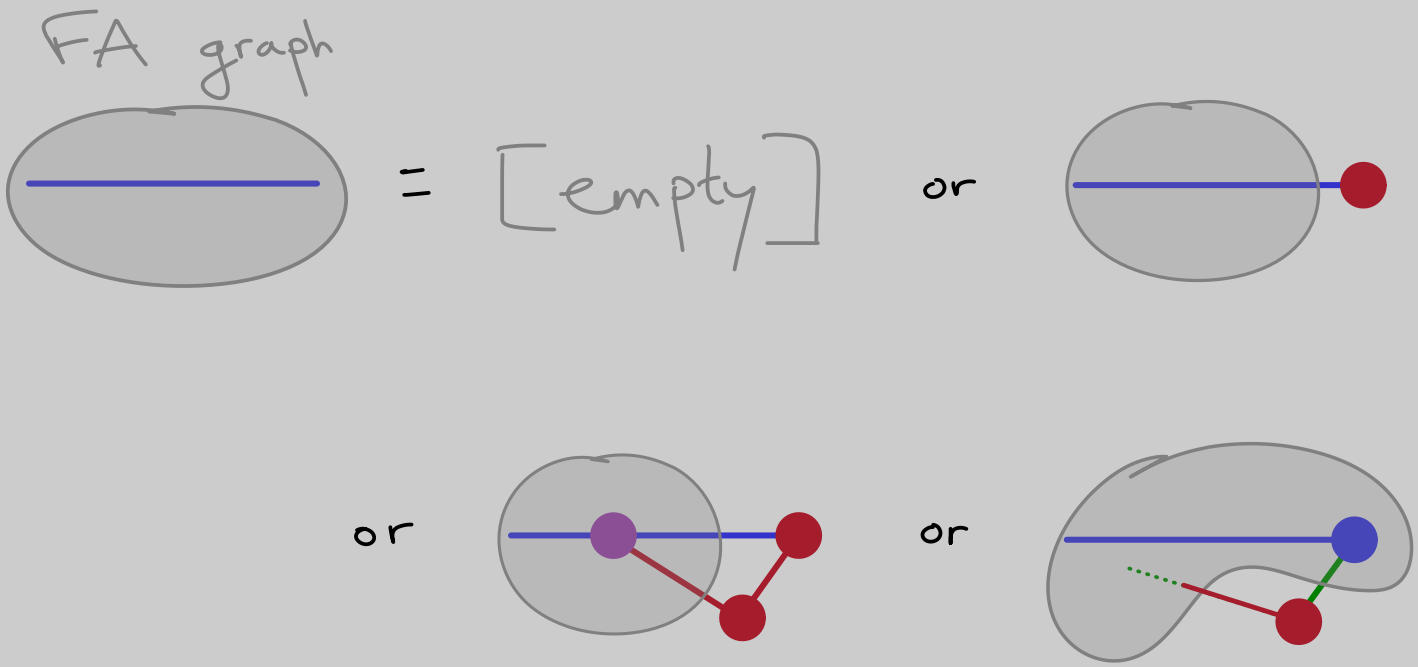


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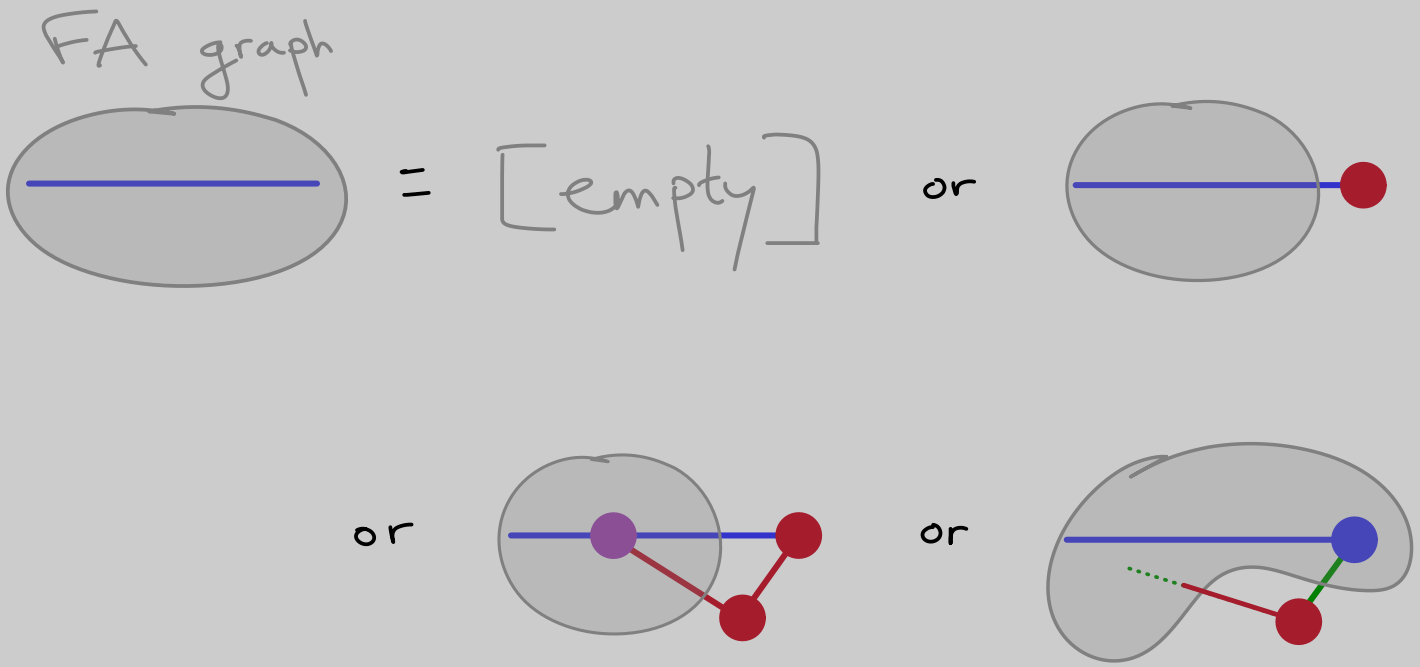
$$\hookrightarrow G(z) = 1 + zG(z) + z^2 \cdot zG'(z) + z \cdot \left( G(z) - zG(z) - 1 \right)$$

# RECURSIVE DECOMPOSITION

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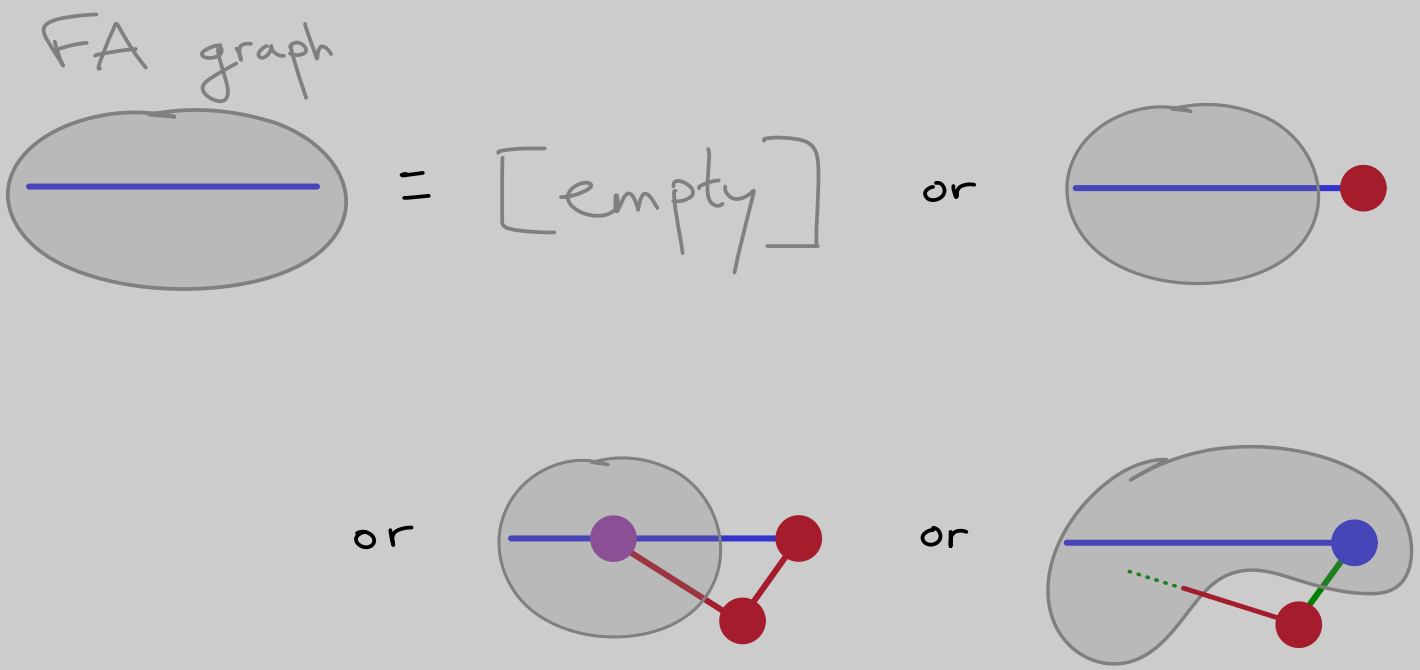


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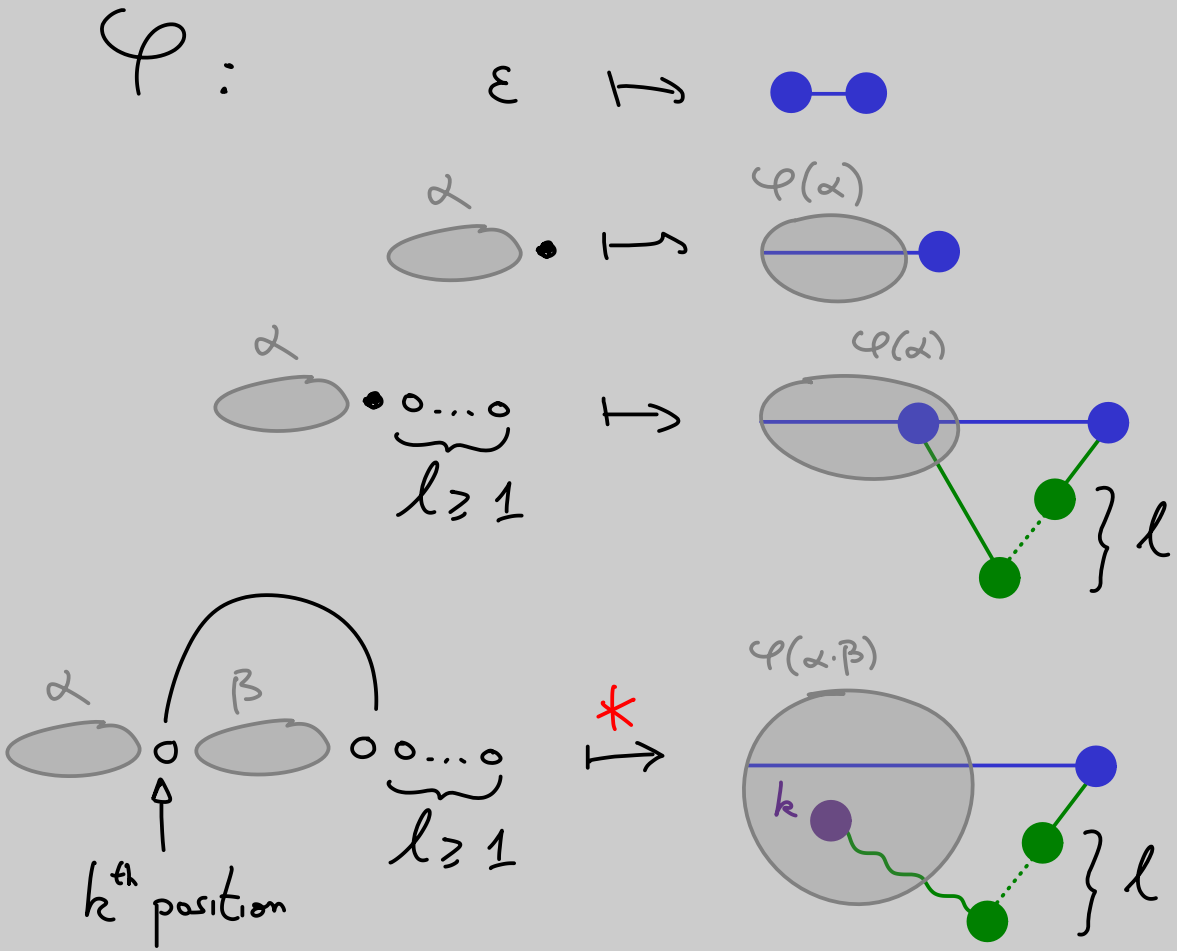
# BIJECTION(S)

## Example



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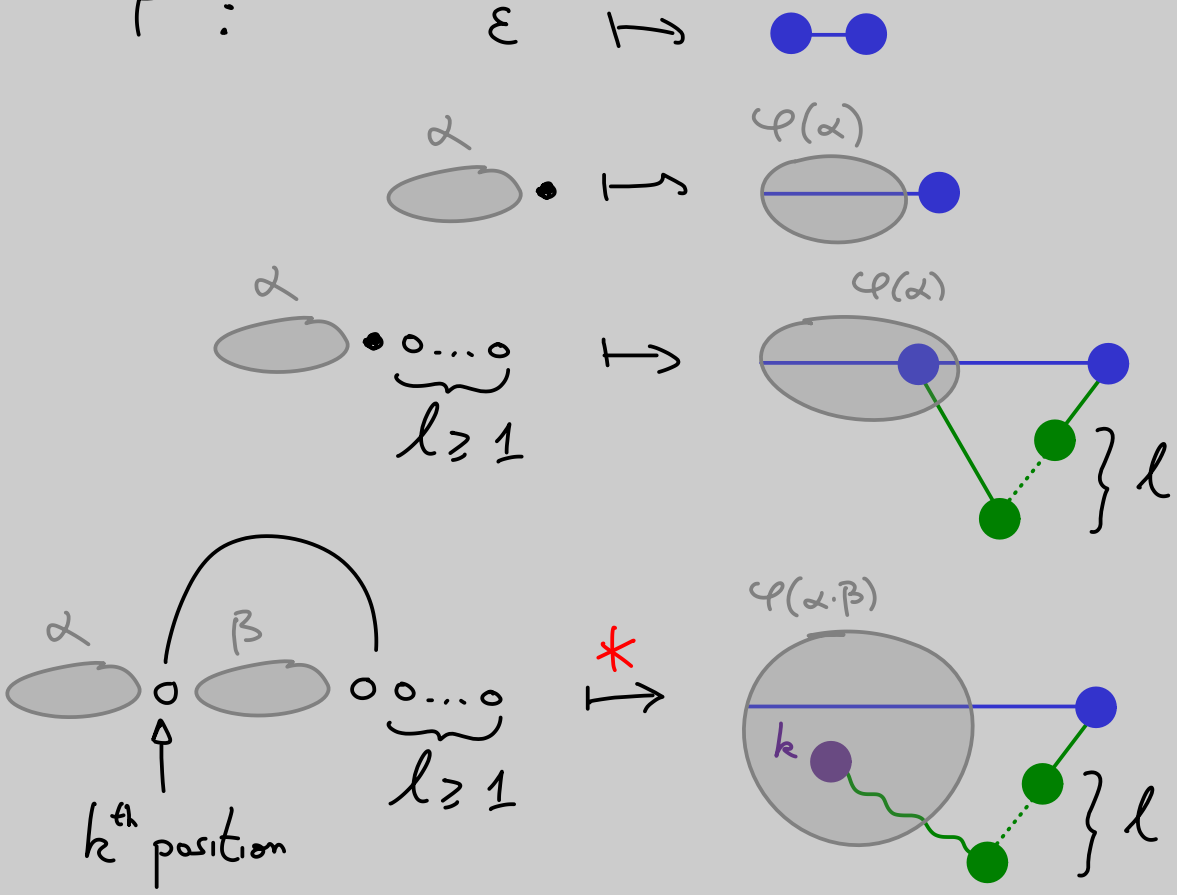
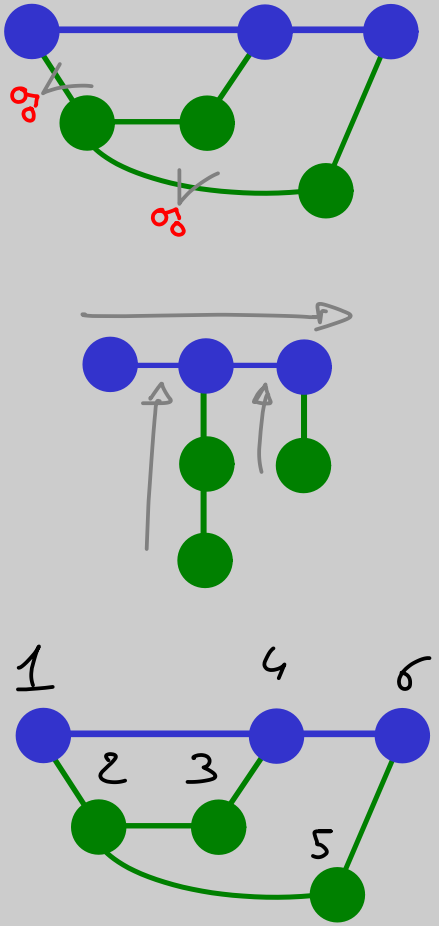



* FA graph vertex labelling :



# BIJECTION(S)

## Open problem

feature branch $\subset$ phoenix $\subset$ fork anywhere

a certain bijection

involutions with 2 types of fixed points

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Ideally, we would want the ?'s to be subclasses defined by pattern avoidance.

# RANDOM GENERATION

Boltzmann sampler for involutions:

$$\forall x > 0, \quad \mathbb{P}_x\left(\begin{array}{c} i \\ \uparrow \\ \text{involution} \end{array}\right) = \frac{x^{|i|}}{|i|! \cdot \hat{G}(x)}$$

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($i$: involution)

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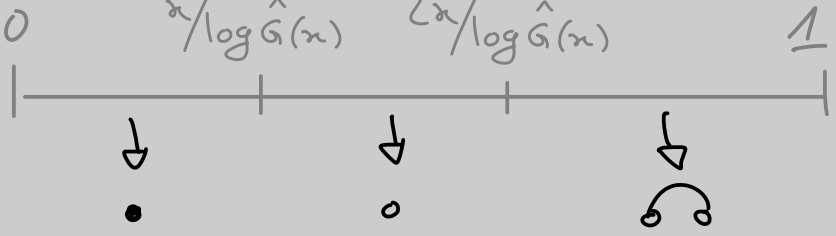
↳ ⋆ size $\rightsquigarrow$ Poisson$\left(2x + \frac{1}{2}x^2\right)$

⋆ draw each of the nodes according to $u \rightsquigarrow \mathcal{U}_{[0,1]}$



# RANDOM GENERATION

Boltzmann → involution with 2 types of fixed points

↓

FA graph ← any working bijection

```
%%time

boltz_involution(10**6)
```

```
CPU times: user 1.58 s, sys: 37.6 ms, total: 1.62 s
Wall time: 1.62 s
```

# ASYMPTOTICS

Cauchy formula: $[z^n]\,\hat{G}(z) \underset{n\to\infty}{\sim} \frac{1}{2i\pi} \oint \frac{\hat{G}(z)}{z^{n+1}}\,dz$

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differences with classic involutions

# OPEN QUESTIONS

* Study other classes of git graphs (realism, ease of computation, ...)

* Evaluate how "close" a typical DAG is to all of these git graph classes

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Thank you :)